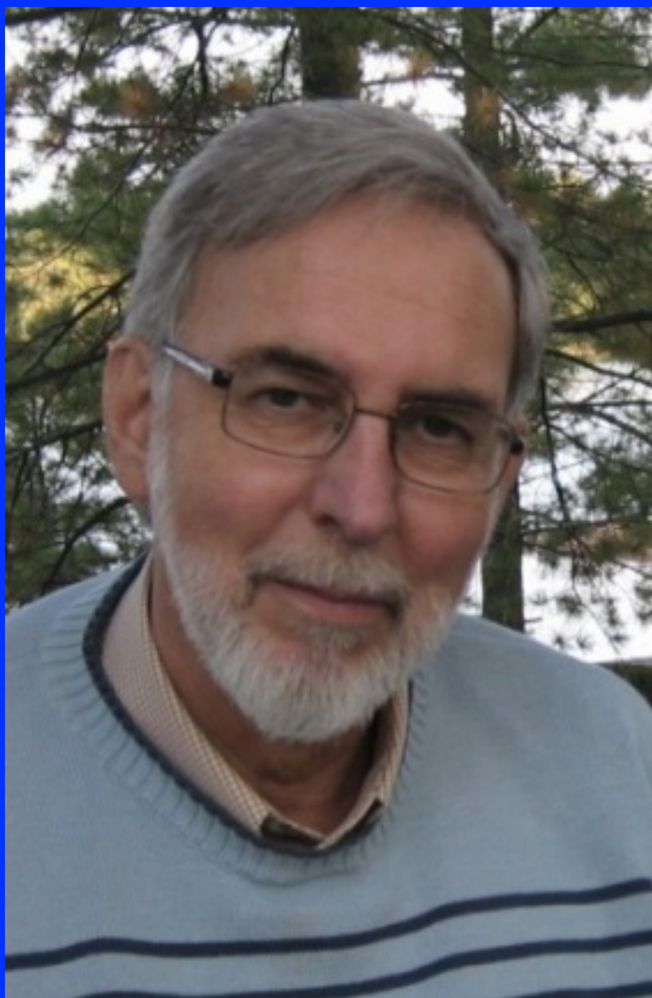
The superfluid mass density



In fondest memory,
Allan Griffin, d. May 19. 2011

Gordon Baym
University of Illinois, Urbana

Finite Temperature Non-Equilibrium Superfluid Systems
Heidelberg, 19 September 2011

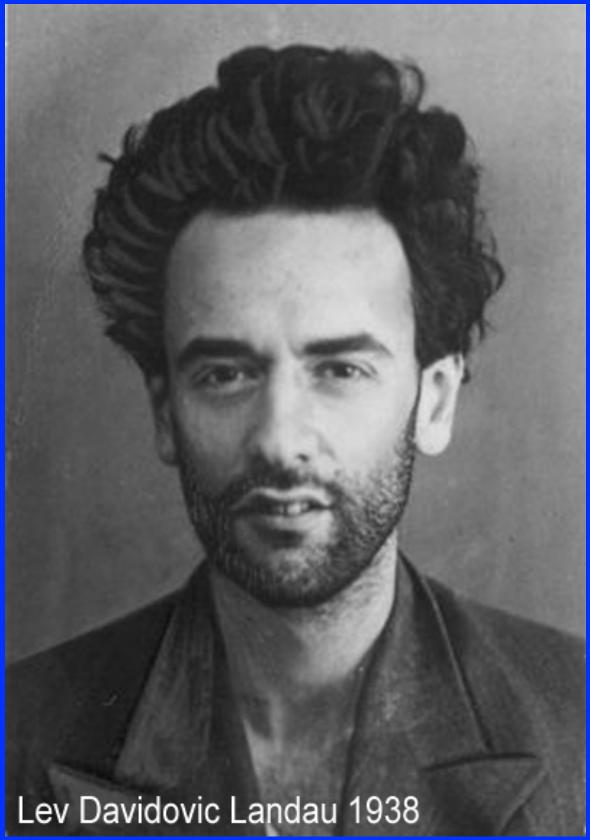
Landau Two-Fluid Model

Can picture superfluid ^{4}He as two interpenetrating fluids:

Normal: density $\rho_n(T)$, velocity v_n

Superfluid: density $\rho_s(T)$, velocity v_s

$$\rho = \rho_n(T) + \rho_s(T)$$



Mass current = $\rho_s v_s + \rho_n v_n$

Entropy current = $s v_n$

:carried by normal fluid only

Second sound (collective mode) =

counter-oscillating normal and superfluids

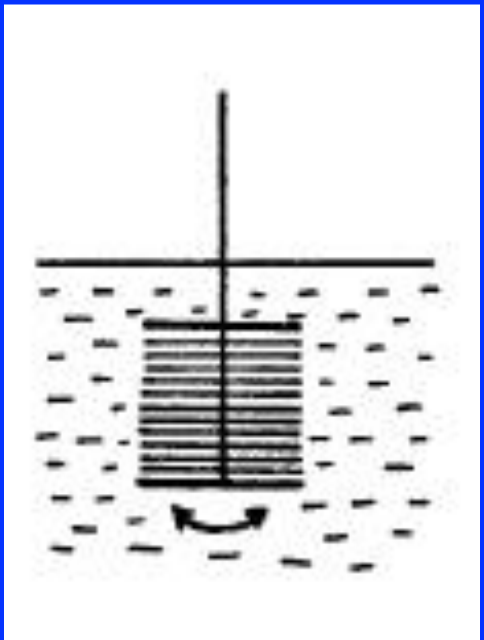
The discovery of superfluidity: a chronology of events in 1935-1938

Allan Griffin
Department of Physics,
University of Toronto
Toronto, Ontario,
Canada M5S 1A7
griffin@physics.utoronto.ca

(September 21, 2006)

The dramatic announcement of superfluidity of liquid ^{4}He in 1938 is one of the defining moments in modern physics. The two short notes which were published back to back in the Jan. 8 issue of *Nature* (by Kapitza [1] working in Moscow and Allen and Misener [2] working in Cambridge University) immediately caught the attention of the physics community. This stimulated feverish activity in the period leading up to World War II, and in the 1950s developed into a major research area called "quantum fluids."

Try to rotate helium slowly. Normal fluid component rotates, but superfluid component stays put.

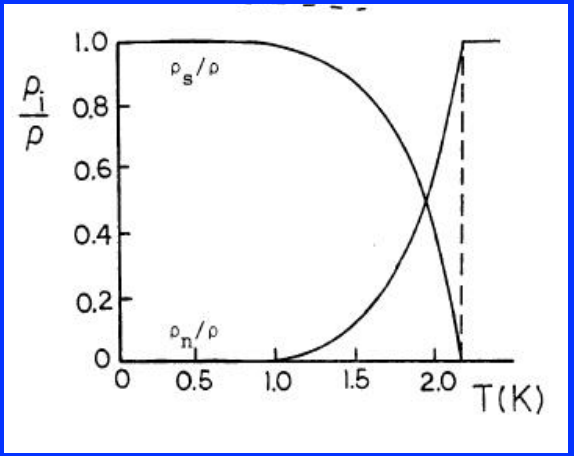


E.L. Andronikashvili, J. Physics, USSR, 1946

Androniskashvili experiment – with stack of closely spaced disks oscillating back and forth – measure how much fluid rotates

Moment of inertia $I = I_{\text{disk}} + I_{\text{fluid}}$

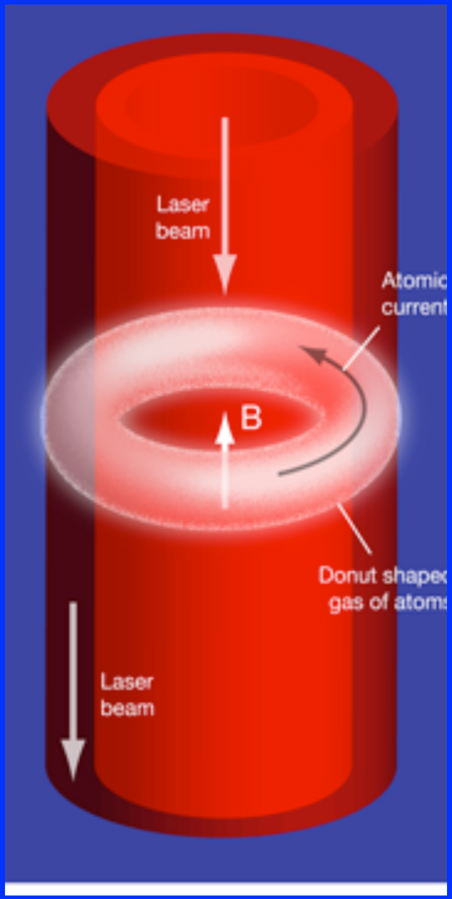
Measure resonant frequency, and deduce I_{fluid} from $$I\frac{d^2\theta}{dt^2} = -k\theta$$



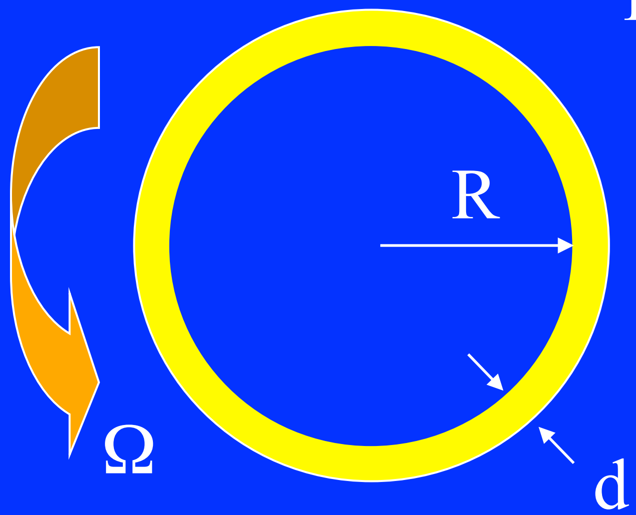
Andronikashvili experiment in cold atoms to measure superfluid mass density

N.R. Cooper and Z. Hadzibabic, PRL 104, 030401 (2010)

Two-photon Raman coupling via beams with different orbital angular momentum difference, to simulate uniform rotation



Hess-Fairbank experiment (*Phys. Rev. Lett. 19, 216 (1967)*)



Rotate thin ($d \ll R$) annulus of liquid ^{4}He at Ω

1) Rotate slowly at $T > T_\lambda$: $\Omega < \Omega_c \sim 1/mR^2$ liquid rotates classically with angular momentum $L = I_{classical}\Omega$.

$$I_{classical} = NmR^2$$

2) Cool to $T < T_\lambda$: liquid rotates with reduced moment of inertia $I(T) < I_{classical}$. $I(T=0) = 0$.

Only the normal fluid rotates. $I(T) = (\rho_n/\rho) I_{classical}$
The superfluid component remains stationary in the lab.

Reduction of moment of inertia is an equilibrium phenomenon.

Moment of inertia of superfluid

Reduction of moment of inertia due condensation

= analog of Meissner effect.

$$I = \frac{\rho_n}{\rho} I_{classical}$$

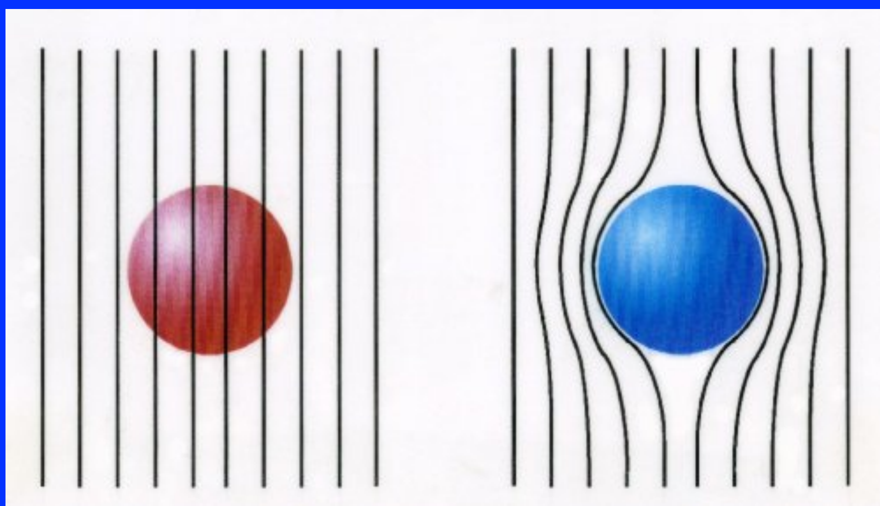
Rotational spectra of nuclei: E = $J^2/2I$, indicate moment of inertia, I, reduced from rigid body value, I_{cl}. Migdal (1959). BCS pairing.

Element	β [7]	x_p	x_n	$\left(\frac{\mathscr{J}}{\mathscr{J}_0}\right)_{rect.}$	$\left(\frac{\mathscr{J}}{\mathscr{J}_0}\right)_{osc.}$	$\left(\frac{\mathscr{J}}{\mathscr{J}_0}\right)_{exper.}$ [7]
Nd^{150}	0.26	0.54	0.94	0.15	0.38	0.35
Sm^{152}	0.24	0.65	1.02	0.17	0.43	0.38
Gd^{154}	0.26	0.52	0.88	0.13	0.35	0.36
Gd^{156}	0.33	0.87	1.37	0.22	0.57	0.48
Gd^{157}	0.29	0.93	1.60	0.22	0.64	0.60
Dy^{162}	0.30	0.84	1.43	0.23	0.57	0.50
Hf^{179}	0.20	0.99	1.75	0.27	0.66	0.52
Os^{186}	0.18	0.44	0.69	0.09	0.26	0.28
Th^{230}	0.22	0.63	0.95	0.15	0.40	0.43
Th^{232}	0.22	0.84	1.42	0.24	0.60	0.44
U^{238}	0.24	0.83	1.29	0.22	0.54	0.43

The Meissner effect

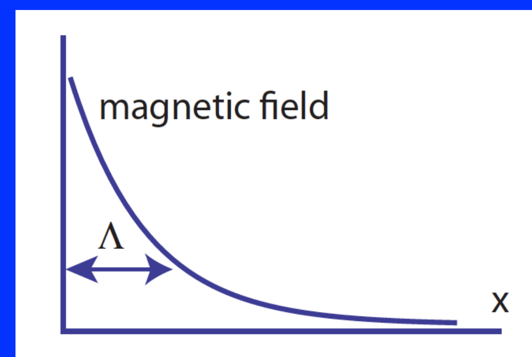
W. Meissner and R. Ochsenfeld, Berlin 1933

Superconductors expel magnetic fields (below critical field):



normal superconducting

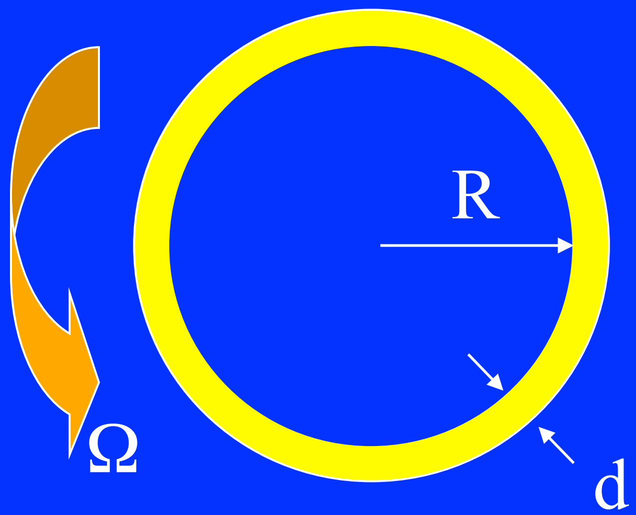
Screening of magnetic field within penetration depth Λ



Fundamental property of superconductors: perfect diamagnets -- not perfect conductors! Equivalent to reduced moment of inertia in neutral superfluids

$$\frac{1}{\Lambda^2} = \frac{4\pi n e^2}{m c^2} \frac{\rho_s}{\rho}$$

Superfluid flow



1) Rotate rapidly at $T>T_\lambda$: $\Omega > \Omega_c$
liquid rotates classically with angular momentum $L=I_{classical}\Omega$.

2) Continue rotating, cool to $T<T_\lambda$:
liquid rotates classically

3) Stop rotation of annulus. Liquid keeps rotating with $L = I_s\Omega$,
where $I_s = (\rho_s/\rho)\, I_{classical}$

Only the superfluid rotates. The normal component is stationary.

Superfluid flow is metastable (albeit with huge lifetime in macroscopic system)

Order parameter of condensate $\Psi(\vec{r}) = |\psi| e^{i\phi(\vec{r})}$

wave function of mode into which particles condense

Defined more rigorously by eigenfunction cf.
largest eigenvalue of density matrix $\langle \psi(\vec{r})\psi^{\dagger}(\vec{r}\,')\rangle \to \Psi(\vec{r})\Psi(\vec{r}\,')^*$

Superfluid velocity: $\vec{v}_s(\vec{r}) = \frac{\hbar}{m}\nabla\phi$

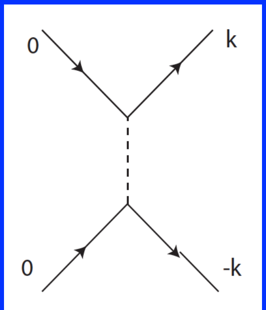
Chemical potential: $\mu = \partial\phi/\partial t$

Superfluid acceleration eqn.: $\frac{\partial \vec{v}_s}{\partial t} + \nabla\mu = 0$

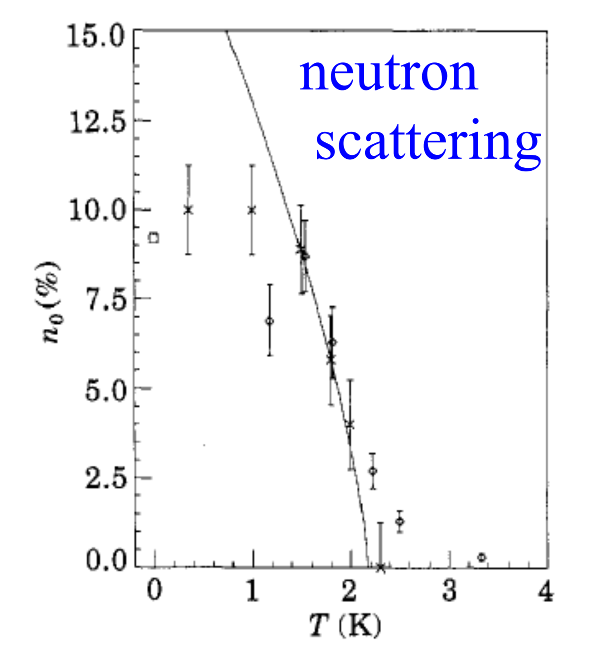
Condensate density is NOT superfluid density

$$\rho_s \neq m|\psi|^2$$

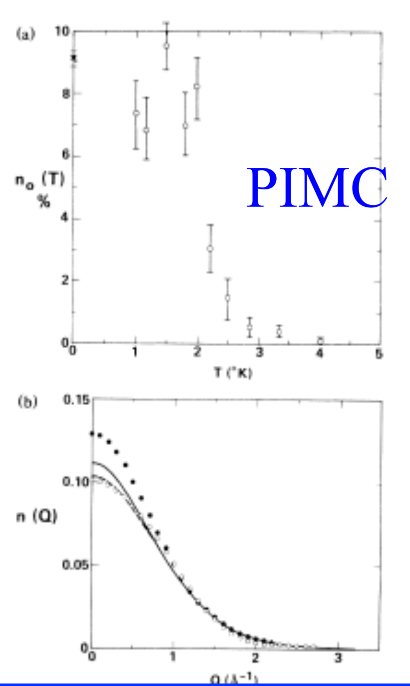
In ground state, interactions drive particles into non-zero momentum single particle states:



In ^{4}He at T=0, $\rho_s/\rho = 1$, while <10% of particles are in condensate



Snow, Wang, & Sokol, Europhys. Lett. 19 (1992)



Ceperley & Pollock, PRL 56 (1986)

Order parameter of BCS paired fermions

Paired seen in amplitude to remove a pair of fermions (↑↓) then add pair back, and come back to same state:

$$\langle \psi_\uparrow^\dagger(1)\psi_\downarrow^\dagger(2)\psi_\downarrow(3)\psi_\uparrow(4)\rangle \simeq \langle \psi_\uparrow^\dagger(1)\psi_\downarrow^\dagger(2)\rangle \, \langle \psi_\downarrow(3)\psi_\uparrow(4)\rangle$$

[Cf., $\langle \psi(\vec{r})\psi^\dagger(\vec{r}\,')\rangle \to \Psi(\vec{r})\Psi(\vec{r}\,')^*$ in Bose system]

Order parameter $\langle \psi_\downarrow(r)\psi_\uparrow(r)\rangle \to \Psi(r)$, as in Bose system

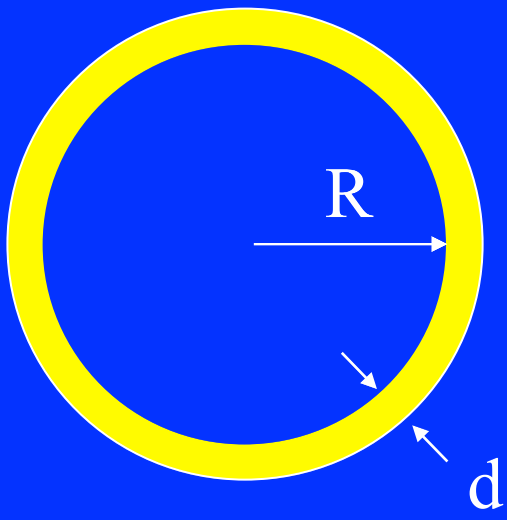
Similar physics as in Bose system

$$\Psi(\vec{r}) = |\psi| e^{i\phi(\vec{r})}$$

Supercurrent velocity: $\vec{v}_s(\vec{r}) = \frac{\hbar}{m}\nabla\phi$

Chemical potential: $\mu = -\partial\phi/\partial t$

(Meta)stability of superfluid flow



Bosons of density n in annulus, T=0

$$H = \frac{\hbar^2}{2m}\nabla\psi^* \cdot \nabla\psi + \frac{g}{2}\psi^*\psi^*\psi\psi$$

Condensates: $\psi_0 = \sqrt{n}$ at rest

$\psi_1 = e^{i\phi}\sqrt{n}$ single vortex

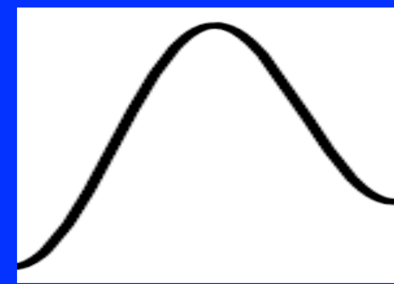
Can one slip continuously from single vortex state to rest state, via condensate $\psi = a\psi_0 + b\psi_1$ with $|a|^2 + |b|^2 = 1$?

Energy density: $$E/V = \frac{\hbar^2}{2mR^2}n|b|^2 + \frac{g}{2}n^2(|a|^4 + |b|^4 + 4|a|^2|b|^2)$$

$$= \frac{\hbar^2}{2mR^2}n|b|^2 + \frac{g}{2}n^2 + 2gn^2|b|^2(1 - |b|^2)$$

Energy density: $= \frac{\hbar^2}{2mR^2} n|b|^2 + \frac{g}{2} n^2 + 2gn^2|b|^2(1-|b|^2)$

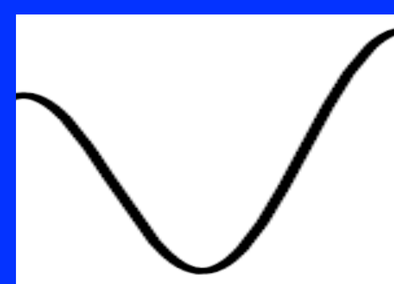
g>0: have barrier of height ~ gnN
for $0 < |b|^2 < 1$



b=0 b=1

Superfluid flow state (vortex) with b=1 is metastable

But for g<0 have minimum

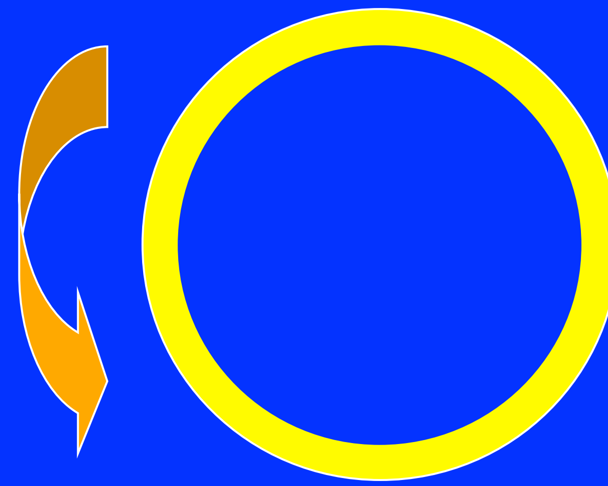


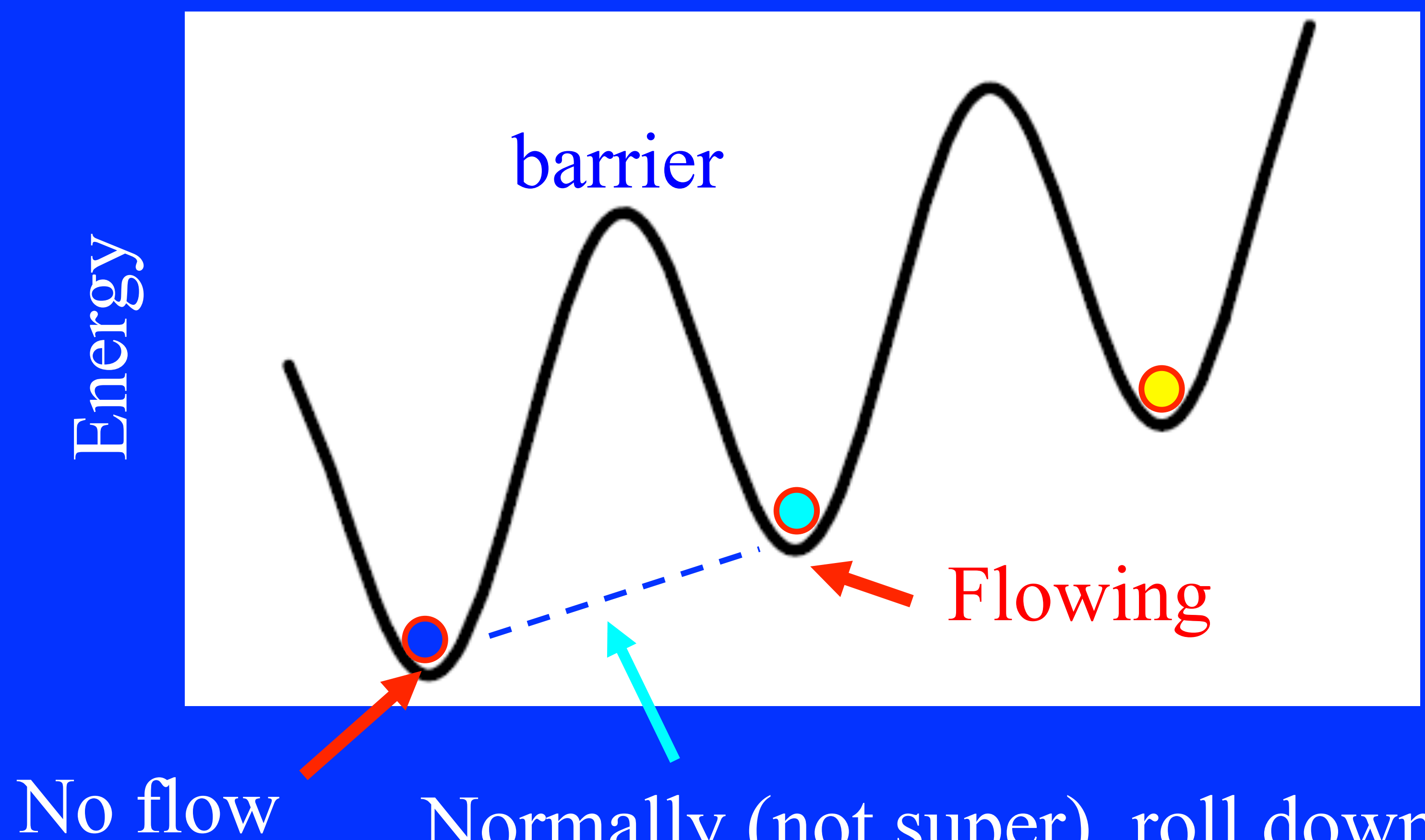
b=0 b=1

Flow is unstable!

Cf. H atom decaying from 2p to 1s state, emitting energy

Superfluid flow difficult to stop because of enormous energy barrier, a hill:





Normally (not super), roll down with no barrier, from flowing state to resting

Condensate density and superfluid mass density

M. Holzmann & GB, Phys. Rev. B 76, 092502 (2007)

For a superfluid flowing down a pipe (at rest) with superfluid velocity v_s in z direction

$$F(v_s, T, \mu) = F(0, T, \mu) + \frac{1}{2}\rho_s v_s^2$$

$$\partial F/\partial v_s = -\langle P_z - M v_s \rangle / V$$

where P_z is the total momentum.

$$\partial^2 F/\partial v_s^2 = \rho - \beta\langle P_z^2 \rangle / V,$$

so that

$$\rho_n = \beta\langle P_z^2 \rangle / V.$$

In normal state, total momentum is Gaussianly distributed:

$\propto \exp(-\beta P^2/2M)$ so that $\beta\langle P_z^2 \rangle / V = M/V$ and $\rho_n = \rho$

In superfluid phase, the total momentum and v_s are entangled and total momentum distribution is not classical.

Exact relation between ρ_s and the condensate density, via the single particle Green's function

P.C. Hohenberg & P.C. Martin, PRL 22 (1963); B.D. B.D. Josephson, PL21 (1966); GB, St. Andrews lectures (1967), A. Griffin PR B30 (1984)

$$G(k,z) = -i\langle T(\psi^\dagger \psi)\rangle(k,z)$$

z = complex frequency

$$\rho_s = -\lim_{k\to 0} \frac{n_0 m^2}{k^2 G(k,0)}$$

Ex. in Bogoliubov mean field (n_0=n),

$$G(k,z) = \frac{z + gn + k^2/2m}{z^2 - gnk^2/m + k^4/4m^2}$$

=> $\rho_s = nm$

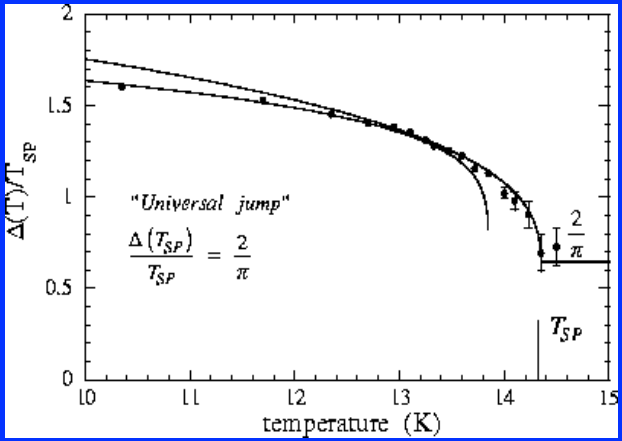
Valid in 2D as well as 3D:

M. Holzmann & GB, PR B 76 (2007);
M. Holzmann, GB, J-P Blaizot,
& F Laloë, PNAS 104 (2007)

In 2D finite size Berestetskii-Kosterlitz-Thouless system,

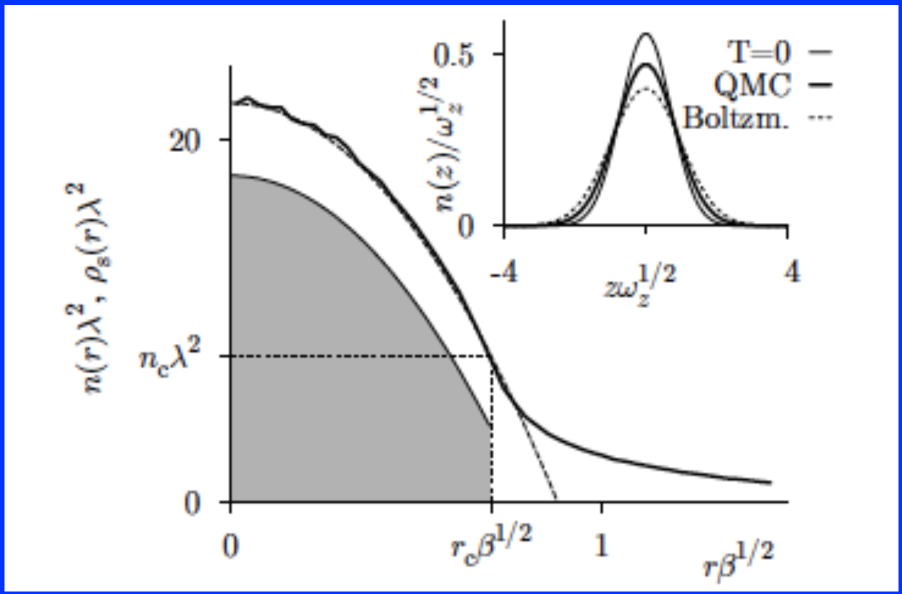
$$n_0 \sim 1/(\text{size})^{2-\eta} \qquad G(k,0) \sim 1/k^{2-\eta}$$

At T_c $\quad \rho_s = \dfrac{m^2 T}{2\pi\eta} \qquad \eta = 1/4$



Order parameter in $CuGeO_3$

Lorenzo et al., EPL. 45 (1999)



Density profile in 2D trap.
Shaded region $\Leftrightarrow \rho_s$

Holtzmann & Krauth, EPL 82 (2008)

Exact (and equivalent) definition of ρ_s in terms of current-current correlation functions

$$\Upsilon_{ij}(\mathbf{r},\mathbf{r}',\omega) = \int dt e^{i\omega(t-t')}\langle[j_i(\mathbf{r},t), j_j(\mathbf{r}',t')]\rangle$$

$$\mathbf{j}(\mathbf{r}) = \frac{1}{2im}[\psi^\dagger(\mathbf{r})\nabla\psi(\mathbf{r}) - \nabla\psi^\dagger(\mathbf{r})\psi(\mathbf{r})]$$

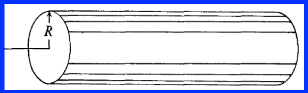
Decompose into longitudinal and transverse components:

$$\Upsilon_{ij}(\mathbf{k},\omega) = \frac{k_i k_j}{k^2}\Upsilon_L(k,\omega) + \left(\delta_{ij} - \frac{k_i k_j}{k^2}\right)\Upsilon_T(k,\omega).$$

f-sum rule => $\rho = \lim_{k\to 0} m^2 \int \frac{d\omega}{2\pi}\frac{\Upsilon_L(k,\omega)}{\omega}$ motion in tube with closed ends

Define normal mass density $\rho_n = \lim_{k\to 0} m^2 \int \frac{d\omega}{2\pi}\frac{\Upsilon_T(k,\omega)}{\omega}$

motion in tube with open ends



$\rho_s = \rho - \rho_n$

How does it work?

At T=0, $$\int \frac{d\omega}{2\pi}\frac{\Upsilon(k,\omega)}{\omega} = 2\sum_{a\neq 0}\frac{|\langle a|j_q|0\rangle|^2}{E_a - E_0}$$

In general, for low-lying states,

$$\langle a|j_k|0\rangle_L \sim k^{1/2} \qquad\qquad E_a - E_0 \sim k$$

and $\rho_n = \rho$. Same for transverse in normal .

In BEC, $\langle phonon|\vec{j}_k|0\rangle \sim k^{1/2}\hat{k}$.

Thus $\langle phonon|\vec{j}_k|0\rangle_T = 0$ and $\rho_n = 0$.

In superconductor with gap, matrix elements vanish in long wavelength limit, while denominators in T integral remain finite and and $\rho_n = 0$.

Meissner effect

The <jj>$_T$ xx correlation gives fundamental description of the Meissner effect as well:

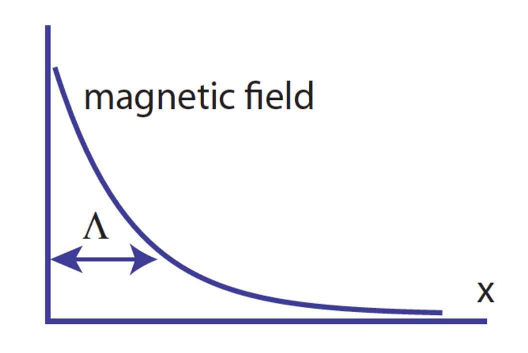
Penetration depth Λ in superconductor

⇔ screening of magnetic field

$$\frac{1}{\Lambda^2} = \frac{4\pi n e^2}{mc^2}\frac{\rho_s}{\rho}$$

$$A_{tot}(k) = \frac{k^2}{k^2 + (1/\Lambda^2)} A(k)$$

=>



Moment of inertia of superfluid

$$I = \frac{\rho_n}{\rho} I_{classical}$$

In terms of <jj>:

$$\mathcal{I}_{ij} = \int \frac{d\mathbf{k}}{(2\pi)^3} \int d\mathbf{r} \int d\mathbf{r}'\, e^{i\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}')} \varepsilon_{isl}\varepsilon_{jmn} r_s r'_m \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \frac{\Upsilon_{ln}(\mathbf{k}\omega)}{\omega}$$

Landau calculation of ρ_s for system w. quasiparticles

$$\langle \boldsymbol{P} \rangle = \sum \boldsymbol{p} \langle N_{\boldsymbol{p}} \rangle$$

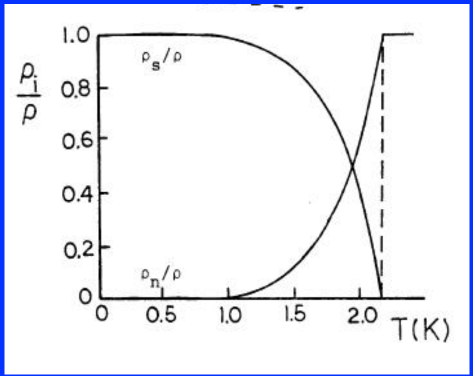
$$\langle N_{\boldsymbol{p}} \rangle = [e^{\beta \{\varepsilon_{\boldsymbol{p}} + \boldsymbol{p} \cdot (\boldsymbol{v}_s - \boldsymbol{v}_n)\}} - 1]^{-1}$$

$$\langle \boldsymbol{P} \rangle = -\sum_{\boldsymbol{p}} \boldsymbol{p}(\boldsymbol{p} \cdot \boldsymbol{v}_n) \frac{\partial}{\partial \varepsilon_{\boldsymbol{p}}} \frac{1}{e^{\beta \varepsilon_{\boldsymbol{p}}} - 1}$$

$$\rho_n = -\int \frac{d\boldsymbol{p}}{(2\pi)^3} \frac{p^2}{3} \frac{\partial}{\partial \varepsilon_{\boldsymbol{p}}} \frac{1}{e^{\beta \varepsilon_{\boldsymbol{p}}} - 1}$$

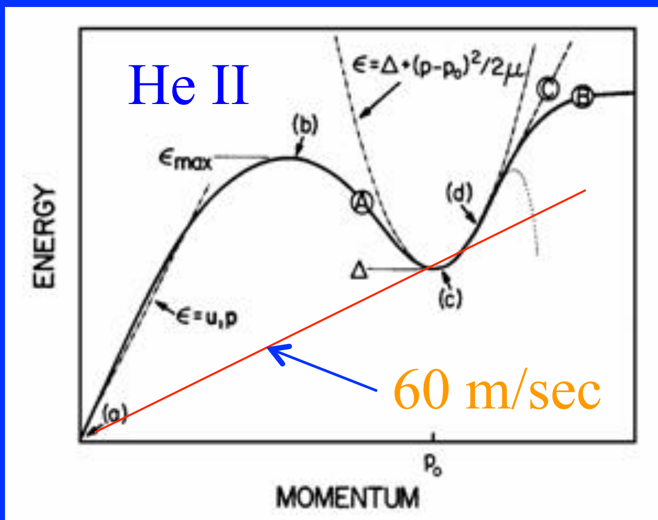
for phonons:

$$\rho_n(T) = \frac{2\pi^2}{45\hbar^3 s^3} T^4$$



The Landau criterion for superfluidity

Superfluid with elementary excitation spectrum $\varepsilon(q)$



Fluid flowing in pipe in x direction, velocity v with respect to walls. In wall frame excitation energy is

$$\varepsilon_v(q) = \varepsilon(q) + vq_x$$

According to Landau:

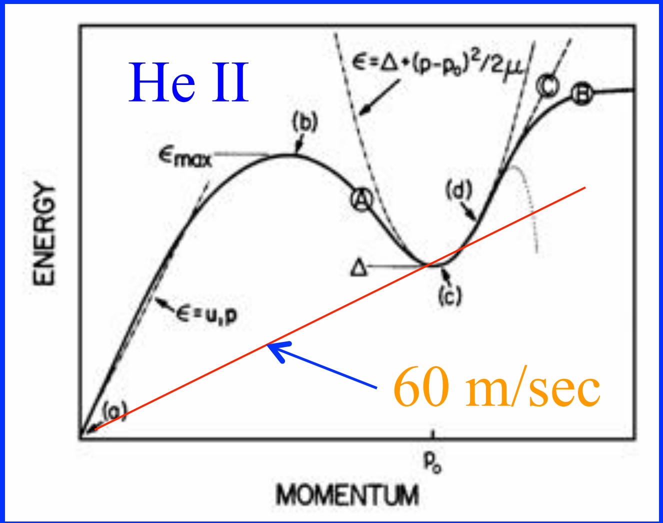
For $v < \varepsilon(q)/q$ cannot make spontaneous excitations (which would decay superflow) and *flow is superfluid*.

For v opposite to q_x and $v > \varepsilon(q)/q$ have $\varepsilon_v(q) < 0$
Can then make excitations spontaneously, and *superfluidity ceases*. v_{crit} = 60 m/sec in superfluid He.

In this way we see that neither phonons nor rotons can be excited if the velocity of flow in helium II is not too large. This means that the flow of the liquid does not slow down, i.e. helium II discloses the phenomenon of superfluidity †. SUFFICIENT

It must be remarked that already the reasons given above are enough to make the superfluidity vanish at sufficiently large velocities. We leave aside the question as to whether superfluidity disappears at smaller velocities for some other reason (the velocity limit obtained from (4.2) is large—the velocity of sound in helium equals 250 m/sec; (4.4) gives a value only several times lower). NECESSARY

L.D. Landau, J. Phys. USSR 5, 71 (1941)



At Landau critical velocity, group and phase velocity of excitations are equal:

$$\frac{\partial \varepsilon}{\partial q} = \frac{\varepsilon}{q}$$

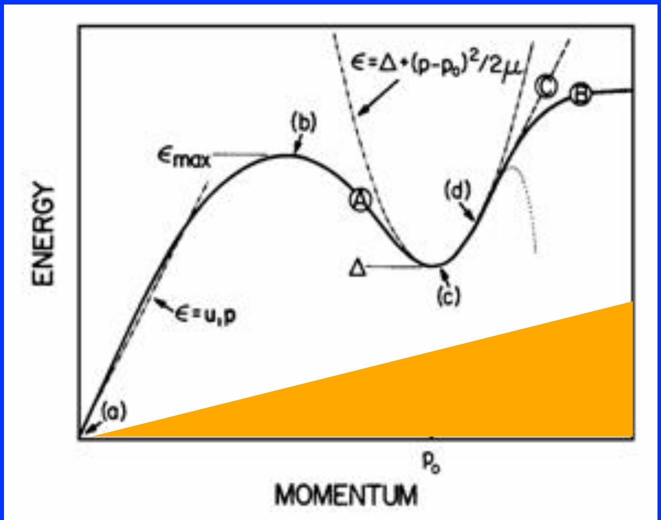
The Landau criterion is neither necessary nor sufficient

Superfluid systems with no “gap”:

1) Dilute solutions of degenerate ^{3}He in superfluid ^{4}He:
Particle-hole spectrum

$$\omega = (\vec{p}+\vec{q})^2/2m - \vec{p}^2/2m$$

reaches down to $\omega = 0$ at $q \neq 0$.

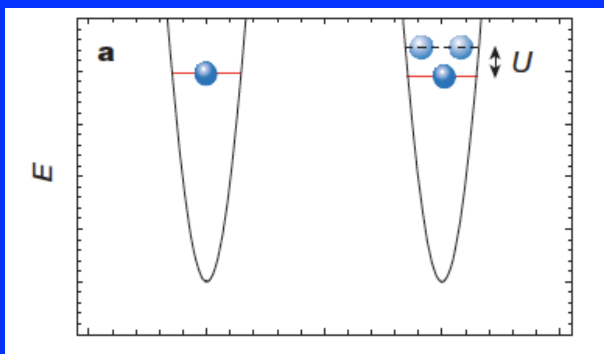


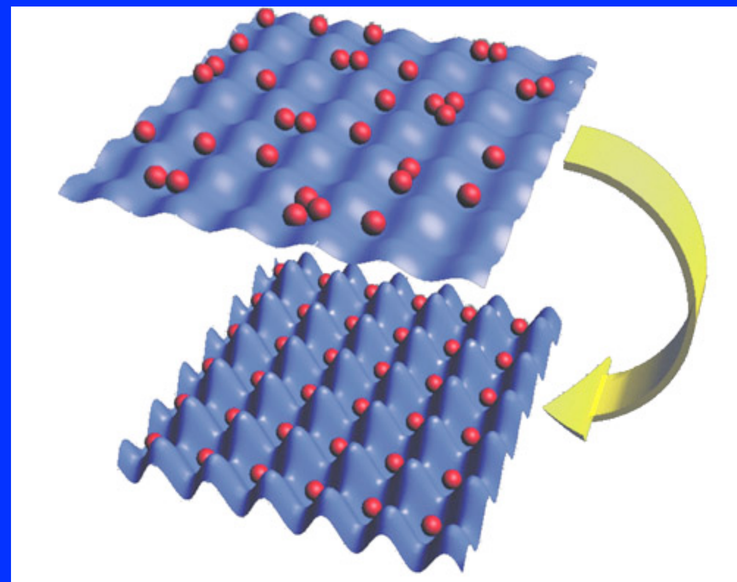
Landau critical velocity vanishes, but system is superfluid.

2) Superfluid ^{4}He at non-zero temperature: Can scatter a phonon of momentum k to –k with zero energy change. Again Landau critical velocity vanishes, but system is a perfectly good superfluid.

Gap also not sufficient to guarantee superfluidity:

ex. bosons in optical lattice:

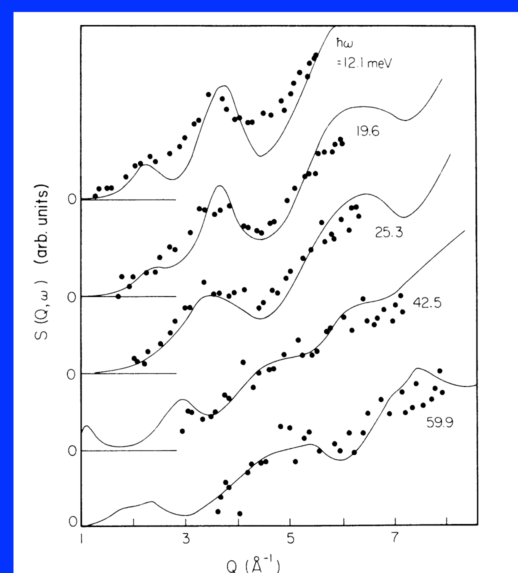




superfluid

Mott insulator

Amorphous solids, e.g., Si doped with H, not superfluid.



What happens when the Landau criterion is violated?

The superfluid mass density becomes less than the total mass density. It does not necessarily vanish!

In dilute solutions of ^{3}He in superfluid ^{4}He,

$$\rho_s = \rho - (m^* - m_3) n_3$$

m^* = ^{3}He effective mass, m_3 = bare mass.

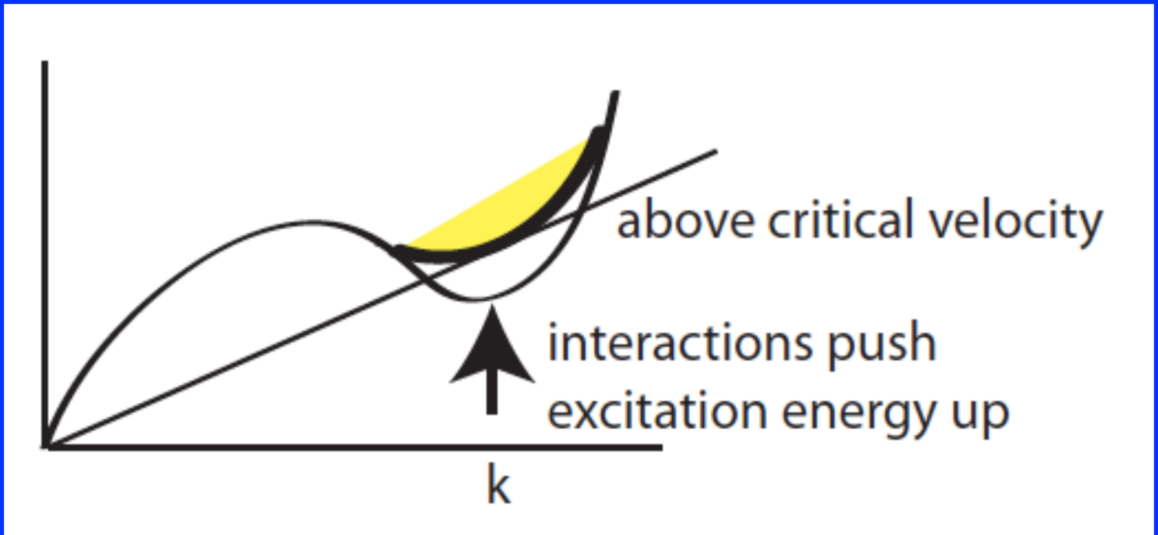
In superfluid ^{4}He at nonzero temperature,

$$\rho_s = \rho - aT^4 - \cdots$$

Formation of non-uniform states

Formation of non-uniform states

L. Pitaevskii, JLTP 87, 127 (1992), GB & CJ Pethick



Beyond the critical velocity spontaneously form excitations of finite momentum k.

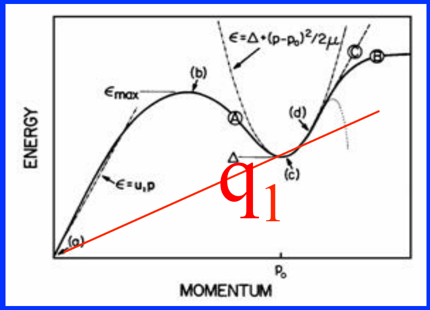
Interactions of these excitations, when repulsive, raises energy of unstable mode k, to make velocity just critical.

Mixing in of modes of momentum k causes condensate to become non-uniform.

Simple model when Landau critical velocity is exceeded

(GB and CJ Pethick)

Weakly interacting Bose gas with finite range interaction g(r), and thus g(q) (> 0) , to produce v_{crit} at non-zero $q = q_1$



In Bogoliubov approx:

$$\varepsilon(q) = \left[\frac{ng(q)q^2}{m} + \left(\frac{q^2}{2m} \right)^2 \right]^{1/2}$$

Critical point: where group velocity = phase velocity,

$$\frac{d(gn)}{d(q^2/2m)} = -1,$$

Study stability of uniform condensate $\psi(x) = \sqrt{n}e^{iqx}$ for $q > q_1$

Since excitations at q_1 can form spontaneously, generate new condensate of form

$$\psi(x) = e^{iqx}\left[\sqrt{n_0} + ue^{-iq_1x} + ve^{iq_1x}\right]$$

Let $u = \zeta\cosh(\phi/2), \qquad v = \zeta\sinh(\phi/2)$

Stable solution $\zeta^2 = (v - v_{crit})\frac{q_1}{2G}$ above critical velocity

$$G = \frac{g_1}{4\varepsilon_1^2}\left[g_1g_2n^2 - \frac{q_1^2}{m}\left(\frac{q_1^2}{m} + 2g_1n\right)\right]$$ = effective repulsion of excitations near q_1

$g_\kappa \equiv g(\kappa q_1)$

Non uniform density: $n(x) = n + 2\sqrt{n}_0\zeta\cosh\phi\cos(q_1x)$

Reduction of superfluid mass density

$$\rho_s = mn - mq_1\frac{\partial\zeta^2}{\partial q} = mn - \frac{q_1^2}{2G}$$

THANK YOU