Kinetic Theory

1. Poisson Brackets:
   (a) Show that for observable $O(p(\mu), q(\mu))$, $dO/dt = \{O, H\}$, along the time trajectory of any micro state $\mu$, where $H$ is the Hamiltonian.
   (b) If the ensemble average $\langle\{O, H\}\rangle = 0$ for any observable $O(p, q)$ in phase space, show that the ensemble density satisfies $\{H, \rho\} = 0$.

2. Equilibrium density: Consider a gas of $N$ particles of mass $m$, in an external potential $U(\vec{q})$. Assume that the one body density $\rho_1(\vec{p}, \vec{q}, t)$, satisfies the Boltzmann equation. For a stationary solution, $\partial \rho_1/\partial t = 0$, it is sufficient from Liouville’s theorem for $\rho_1$ to satisfy $\rho_1 \propto \exp[-\beta (p^2/2m + U(\vec{q}))]$. Prove that this condition is also necessary by using the H-theorem as follows.
   (a) Find $\rho_1(\vec{p}, \vec{q})$ that minimizes $H = N \int d^3\vec{p}d^3\vec{q} \rho_1(\vec{p}, \vec{q}) \ln \rho_1(\vec{p}, \vec{q})$, subject to the constraint that the total energy $E = \langle H \rangle$ is constant. (Hint: Use the method of Lagrange multipliers to impose the constraint.)
   (b) For a mixture of two gases (particles of masses $m_a$ and $m_b$) find the distributions $\rho_1^{(a)}$ and $\rho_1^{(b)}$ that minimize $H = H^{(a)} + H^{(b)}$ subject to the constraint of constant total energy. Hence show that the kinetic energy per particle can serve as an empirical temperature.

3. Evolving a canonical harmonic oscillator density: A dilute gas of non-interacting particles is in equilibrium in a harmonic potential, such that the density for each particle has the form
   \[ \rho_0(\vec{q}, \vec{p}) = \exp \left[ -\beta \left( \frac{Kq^2}{2} + \frac{p^2}{2m} \right) \right] \left( \frac{\beta}{2\pi} \right)^3 \left( \frac{K}{m} \right)^{3/2}. \]
   At time $t = 0$, and external force $\vec{F}(t)$ is applied, changing the one particle Hamiltonian to $H_0 - \vec{q} \cdot \vec{F}(t)$.
   (a) Write down the (Liouville) equation governing subsequent evolution of the one particle density.
   (b) Confirm that the density at later times satisfies $\rho(\vec{q}, \vec{p}, t) = \rho_0(\vec{q} - \langle \vec{q} \rangle_t, \vec{p} - \langle \vec{p} \rangle_t)$, and find the equations of motion for $\langle \vec{q} \rangle_t$ and $\langle \vec{p} \rangle_t$. 
(c) Compute the entropy $S(t)$ associated with the probability density $\rho$.

(d) Would a similar time dependent shift of the density work in the case of the canonical weight associated with a general potential $V(\mathbf{q})$ (e.g. $V(\mathbf{q}) \propto q^4$) driven by an external force?

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4. Gas mixture: Consider a mixture of two gases (a) and (b), in a box of volume $V$.

(a) Write down the Boltzmann equations for the one particle densities $f_1^{(a)}$, and, $f_1^{(b)}$, in terms of the Liouville operators $L_x \equiv [\partial_t + (\mathbf{p}/m_x) \cdot \nabla]$, and collision operators

$$C_{x,y} = -\int d^3\mathbf{p}_1 d^2\Omega \left| \vec{v}_1 - \vec{v}_2 \right| \left[f_1^{(x)}(\mathbf{p}_1, \mathbf{q}_1) f_1^{(y)}(\mathbf{p}_2, \mathbf{q}_1) - f_1^{(x)}(\mathbf{p}_1', \mathbf{q}_1) f_1^{(y)}(\mathbf{p}_2', \mathbf{q}_1) \right],$$

where $x = a, b$ and $y = a, b$.

(b) Assuming that the collision terms are much more dominant than the Liouville drifts, write down a zeroth order solution to the Boltzmann equations.

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5. Zeroth-order hydrodynamics: The hydrodynamic equations resulting from the conservation of particle number, momentum, and energy in collisions are (in a uniform box):

$$\begin{cases}
\partial_t n + \partial_\alpha (nu_\alpha) = 0, \\
\partial_t u_\alpha + u_\beta \partial_\beta u_\alpha = -\frac{1}{mn} \partial_\beta P_{\alpha\beta}, \\
\partial_t \varepsilon + u_\alpha \partial_\alpha \varepsilon = -\frac{1}{n} \partial_\alpha h_\alpha - \frac{1}{n} P_{\alpha\beta} u_{\alpha\beta},
\end{cases}$$

where $n$ is the local density, $\vec{u} = \langle \mathbf{p}/m \rangle$, $u_{\alpha\beta} = (\partial_\alpha u_\beta + \partial_\beta u_\alpha)/2$, and $\varepsilon = \langle mc^2/2 \rangle$, with $\vec{c} = \vec{p}/m - \vec{u}$.

(a) For the zeroth order density

$$f_1^0(\mathbf{p}, \mathbf{q}, t) = \frac{n(\mathbf{q}, t)}{(2\pi mk_B T(\mathbf{q}, t))^{3/2}} \exp \left[ -\frac{(\mathbf{p} - m\vec{u}(\mathbf{q}, t))^2}{2mk_B T(\mathbf{q}, t)} \right],$$

calculate the pressure tensor $P_{\alpha\beta}^0 = mn \langle c_\alpha c_\beta \rangle^0$, and the heat flux $h_\alpha^0 = nm \langle c_\alpha c^2/2 \rangle^0$.

(b) Obtain the zeroth order hydrodynamic equations governing the evolution of $n(\mathbf{q}, t)$, $\vec{u}(\mathbf{q}, t)$, and $T(\mathbf{q}, t)$. 
(c) Show that the above equations imply \(D_t \ln \left( nT^{-3/2} \right) = 0\), where \(D_t = \partial_t + u_\beta \partial_\beta\) is the material derivative along streamlines.

(d) Write down the expression for the function \(H^0(t) = \int d^3q d^3p f^0_1(\vec{p}, \vec{q}, t) \ln f^0_1(\vec{p}, \vec{q}, t)\), after performing the integrations over \(\vec{p}\), in terms of \(n(\vec{q}, t), \vec{u}(\vec{q}, t), \) and \(T(\vec{q}, t)\).

(e) Using the hydrodynamic equations in (b) calculate \(dH^0/dt\).

(f) Discuss the implications of the result in (e) for approach to equilibrium.

6. Viscosity: Consider a classical gas between two plates separated by a distance \(w\). One plate at \(y = 0\) is stationary, while the other at \(y = w\) moves with a constant velocity \(v_x = u\). A zeroth order approximation to the one particle density is,

\[
f^0_1(\vec{p}, \vec{q}) = \frac{n}{(2\pi mk_BT)^{3/2}} \exp \left[ -\frac{1}{2mk_BT} \left( (p_x - m\alpha y)^2 + p_y^2 + p_z^2 \right) \right],
\]

obtained from the uniform Maxwell–Boltzmann distribution by substituting the average value of the velocity at each point. (\(\alpha = u/w\) is the velocity gradient.)

(a) The above approximation does not satisfy the Boltzmann equation as the collision term vanishes, while \(df^0_1/dt \neq 0\). Find a better approximation, \(f^1_1(\vec{p})\), by linearizing the Boltzmann equation, in the single collision time approximation, to

\[
\mathcal{L} \left[ f^1_1 \right] \approx \left[ \frac{\partial}{\partial t} + \frac{\vec{p}}{m} \cdot \frac{\partial}{\partial \vec{q}} \right] f^0_1 \approx -\frac{f^1_1 - f^0_1}{\tau_x},
\]

where \(\tau_x\) is a characteristic mean time between collisions.

(b) Calculate the net transfer \(\Pi_{xy}\) of the \(x\) component of the momentum, of particles passing through a plane at \(y\), per unit area and in unit time.

(c) Note that the answer to (b) is independent of \(y\), indicating a uniform transverse force \(F_x = -\Pi_{xy}\), exerted by the gas on each plate. Find the coefficient of viscosity, defined by \(\eta = F_x/\alpha\).

7. (Optional) Light and matter: In this problem we use kinetic theory to explore the equilibrium between atoms and radiation.

(a) The atoms are assumed to be either in their ground state \(a_0\), or in an excited state \(a_1\), which has a higher energy \(\varepsilon\). By considering the atoms as a collection of \(N\) fixed two-state
systems of energy $E$ (i.e. ignoring their coordinates and momenta), calculate the ratio $n_1/n_0$ of densities of atoms in the two states as a function of temperature $T$.

Consider photons $\gamma$ of frequency $\omega = \varepsilon/k$ and momentum $|\vec{p}| = \hbar \omega/c$, which can interact with the atoms through the following processes:

(i) *Spontaneous emission:* $a_1 \to a_0 + \gamma$.

(ii) *Adsorption:* $a_0 + \gamma \to a_1$.

(iii) *Stimulated emission:* $a_1 + \gamma \to a_0 + \gamma + \gamma$.

Assume that spontaneous emission occurs with a probability $\sigma_{sp}$, and that adsorption and stimulated emission have corresponding constant (angle-independent) probabilities (cross-sections) of $\sigma_{ad}$ and $\sigma_{st}$, respectively.

(b) Write down the Boltzmann equation governing the density $f$ of the photon gas, treating the atoms as fixed scatterers of densities $n_0$ and $n_1$.

(c) Find the equilibrium density $f_{eq}$ for the photons of the above frequency.

(d) According to Planck’s law, the density of photons at a temperature $T$ depends on their frequency $\omega$ as $f_{eq} = \left[\exp(\hbar \omega/k_B T) - 1\right]^{-1}/\hbar^3$. What does this imply about the above cross sections?

(e) Consider a situation in which light shines along the $x$ axis on a collection of atoms whose boundary coincides with the $x = 0$ plane, as illustrated in the figure.

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\[ \gamma \]
\[ \vdots \]
\[ \vdots \]
\[ \vdots \]
\[ \text{vacuum} \]
\[ \text{matter} (n_0, n_1) \]
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Clearly, $f$ will depend on $x$ (and $p_x$), but will be independent of $y$ and $z$. Adapt the Boltzmann equation you propose in part (b) to the case of a uniform incoming flux of photons with momentum $\vec{p} = \hbar \omega \hat{x}/c$. What is the penetration length across which the incoming flux decays?

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8. **Effusion:** The probability distribution for speed $c$ of particles of mass $m$ in a gas at temperature $T$ is proportional to $c^2 e^{-\frac{c^2}{2\sigma^2}}$, with $\sigma^2 = k_B T/m$. Some particles are allowed to leak (effuse) out of a small hole with diameter much less than the mean free path.
(a) Show that the probability distribution for speed of the escaping particles is proportional to $c^3 e^{-\frac{c^2}{2\sigma^2}}$.

(b) Find the average kinetic energy of the escaping particles.

(c) What is the fraction of escaping particles with kinetic energy greater than $E$?

† Reviewing the problems and solutions provided on the course web-page for preparation for Test 2 should help you with the above problems.