1. Poisson Brackets:
   (a) Show that for observable $\mathcal{O}(p(\mu), q(\mu))$, $d\mathcal{O}/dt = \{\mathcal{O}, \mathcal{H}\}$, along the time trajectory of any micro state $\mu$, where $\mathcal{H}$ is the Hamiltonian.
   
   (b) If the ensemble average $\langle\{\mathcal{O}, \mathcal{H}\}\rangle = 0$ for any observable $\mathcal{O}(p, q)$ in phase space, show that the ensemble density satisfies $\{\mathcal{H}, \rho\} = 0$.

2. Equilibrium density: Consider a gas of $N$ particles of mass $m$, in an external potential $U(\vec{q})$. Assume that the one body density $\rho_1(\vec{p}, \vec{q}, t)$, satisfies the Boltzmann equation. For a stationary solution, $\partial \rho_1/\partial t = 0$, it is sufficient from Liouville’s theorem for $\rho_1$ to satisfy $\rho_1 \propto \exp \left[ -\beta \left( \frac{p^2}{2m} + U(\vec{q}) \right) \right]$. Prove that this condition is also necessary by using the H-theorem as follows.
   
   (a) Find $\rho_1(\vec{p}, \vec{q})$ that minimizes $H = N \int d^3\vec{p}d^3\vec{q} \rho_1(\vec{p}, \vec{q}) \ln \rho_1(\vec{p}, \vec{q})$, subject to the constraint that the total energy $E = \langle H \rangle$ is constant. (Hint: Use the method of Lagrange multipliers to impose the constraint.)
   
   (b) For a mixture of two gases (particles of masses $m_a$ and $m_b$) find the distributions $\rho_1^{(a)}$ and $\rho_1^{(b)}$ that minimize $H = H^{(a)} + H^{(b)}$ subject to the constraint of constant total energy. Hence show that the kinetic energy per particle can serve as an empirical temperature.

3. Evolving a canonical harmonic oscillator density: A dilute gas of non-interacting particles is in equilibrium in a harmonic potential, such that the density for each particle has the form
   
   $\rho_0(\vec{q}, \vec{p}) = \exp \left[ -\beta \left( \frac{Kq^2}{2} + \frac{p^2}{2m} \right) \right] \left( \frac{\beta}{2\pi} \right)^3 \left( \frac{K}{m} \right)^{3/2}$.

   At time $t = 0$, and external force $\vec{F}(t)$ is applied, changing the one particle Hamiltonian to $H_0 - \vec{q} \cdot \vec{F}(t)$.
   
   (a) Write down the (Liouville) equation governing subsequent evolution of the one particle density.
   
   (b) Confirm that the density at later times satisfies, $\rho(\vec{q}, \vec{p}, t) = \rho_0(\vec{q} - \langle \vec{q} \rangle_t, \vec{p} - \langle \vec{p} \rangle_t)$, and find the equations of motion for $\langle \vec{q} \rangle_t$ and $\langle \vec{p} \rangle_t$. 
(c) Compute the entropy $S(t)$ associated with the probability density $\rho$.

(d) Would a similar time dependent shift of the density work in the case of the canonical weight associated with a general potential $\mathcal{V}(\vec{q})$ (e.g. $\mathcal{V}(\vec{q}) \propto q^4$) driven by an external force?

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4. Gas mixture: Consider a mixture of two gases (a) and (b), in a box of volume $V$.

(a) Write down the Boltzmann equations for the one particle densities $f_1^{(a)}$ and $f_1^{(b)}$, in terms of the Liouville operators $L_x \equiv [\partial_t + (\vec{p}/m_x) \cdot \nabla]$, and collision operators

$$C_{x,y} = -\int d^3 \vec{p}_2 d^2 \Omega \left| \bar{v}_1 - \bar{v}_2 \right| \left[ f_1^{(x)}(\vec{p}_1, \vec{q}_1) f_1^{(y)}(\vec{p}_2, \vec{q}_1) - f_1^{(x)}(\vec{p}_1', \vec{q}_1) f_1^{(y)}(\vec{p}_2', \vec{q}_1) \right],$$

where $x = a, b$ and $y = a, b$.

(b) Assuming that the collision terms are much more dominant than the Liouville drifts, write down a zeroth order solution to the Boltzmann equations.

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5. Zeroth-order hydrodynamics: The hydrodynamic equations resulting from the conservation of particle number, momentum, and energy in collisions are (in a uniform box):

$$\begin{align*}
\partial_t n + \partial_\alpha (nu_\alpha) &= 0 \\
\partial_t u_\alpha + u_\beta \partial_\beta u_\alpha &= -\frac{1}{mn} \partial_\beta P_{\alpha\beta} \\
\partial_t \varepsilon + u_\alpha \partial_\alpha \varepsilon &= -\frac{1}{n} \partial_\alpha h_\alpha - \frac{1}{n} P_{\alpha\beta} u_{\alpha\beta}
\end{align*}$$

where $n$ is the local density, $\bar{u} = \langle \vec{p}/m \rangle$, $u_{\alpha\beta} = (\partial_\alpha u_\beta + \partial_\beta u_\alpha)/2$, and $\varepsilon = \langle mc^2/2 \rangle$, with $\bar{c} = \vec{p}/m - \bar{u}$.

(a) For the zeroth order density

$$f_1^0(\vec{p}, \vec{q}, t) = \frac{n(\vec{q}, t)}{(2\pi mk_B T(\vec{q}, t))^{3/2}} \exp \left[ -\frac{(\vec{p} - m\bar{u}(\vec{q}, t))^2}{2mk_B T(\vec{q}, t)} \right],$$

calculate the pressure tensor $P_{\alpha\beta}^0 = mn \langle c_\alpha c_\beta \rangle^0$, and the heat flux $h_\alpha^0 = nm \langle c_\alpha c^2/2 \rangle^0$.

(b) Obtain the zeroth order hydrodynamic equations governing the evolution of $n(\vec{q}, t)$, $\bar{u}(\vec{q}, t)$, and $T(\vec{q}, t)$.
(c) Show that the above equations imply
\[ D_t \ln \left( nT^{-3/2} \right) = 0, \]
where \( D_t = \partial_t + u_\beta \partial_\beta \) is the material derivative along streamlines.

(d) Write down the expression for the function
\[ H^0(t) = \int d^3\vec{\tilde{q}}d^3\tilde{p} f^0_1(\tilde{p}, \tilde{q}, t) \ln f^0_1(\tilde{p}, \tilde{q}, t), \]
after performing the integrations over \( \tilde{p} \), in terms of \( n(\tilde{q}, t), \tilde{u}(\tilde{q}, t), \) and \( T(\tilde{q}, t) \).

(e) Using the hydrodynamic equations in (b) calculate \( dH^0/dt \).

(f) Discuss the implications of the result in (e) for approach to equilibrium.

6. **Viscosity:** Consider a classical gas between two plates separated by a distance \( w \). One plate at \( y = 0 \) is stationary, while the other at \( y = w \) moves with a constant velocity \( v_\sigma = u \). A zeroth order approximation to the one particle density is,
\[ f^0_1(\tilde{p}, \tilde{q}) = \frac{n}{(2\pi mk_B T)^{3/2}} \exp \left[ -\frac{1}{2mk_B T} \left( (p_x - m\alpha y)^2 + p_y^2 + p_z^2 \right) \right], \]

obtained from the uniform Maxwell–Boltzmann distribution by substituting the average value of the velocity at each point. \( \alpha = u/w \) is the velocity gradient.)

(a) The above approximation does not satisfy the Boltzmann equation as the collision term vanishes, while \( df^0_1/dt \neq 0 \). Find a better approximation, \( f^1_1(\tilde{p}) \), by linearizing the Boltzmann equation, in the single collision time approximation, to
\[ \mathcal{L} [f^1_1] \approx \left[ \frac{\partial}{\partial t} + \frac{\tilde{p}}{m} \cdot \frac{\partial}{\partial \tilde{q}} \right] f^0_1 \approx -\frac{f^1_1 - f^0_1}{\tau_x}, \]

where \( \tau_x \) is a characteristic mean time between collisions.

(b) Calculate the net transfer \( \Pi_{xy} \) of the \( x \) component of the momentum, of particles passing through a plane at \( y \), per unit area and in unit time.

(c) Note that the answer to (b) is independent of \( y \), indicating a uniform transverse force \( F_x = -\Pi_{xy} \), exerted by the gas on each plate. Find the coefficient of viscosity, defined by \( \eta = F_x/\alpha \).

7. **(Optional) Light and matter:** In this problem we use kinetic theory to explore the equilibrium between atoms and radiation.

(a) The atoms are assumed to be either in their ground state \( a_0 \), or in an excited state \( a_1 \), which has a higher energy \( \varepsilon \). By considering the atoms as a collection of \( N \) fixed two-state
systems of energy $E$ (i.e. ignoring their coordinates and momenta), calculate the ratio $n_1/n_0$ of densities of atoms in the two states as a function of temperature $T$.

Consider photons $\gamma$ of frequency $\omega = \varepsilon/\hbar$ and momentum $|\vec{p}| = \hbar\omega/c$, which can interact with the atoms through the following processes:

(i) *Spontaneous emission*: $a_1 \rightarrow a_0 + \gamma$.
(ii) *Adsorption*: $a_0 + \gamma \rightarrow a_1$.
(iii) *Stimulated emission*: $a_1 + \gamma \rightarrow a_0 + \gamma + \gamma$.

Assume that spontaneous emission occurs with a probability $\sigma_{sp}$, and that adsorption and stimulated emission have corresponding constant (angle-independent) probabilities (cross-sections) of $\sigma_{ad}$ and $\sigma_{st}$, respectively.

(b) Write down the Boltzmann equation governing the density $f$ of the photon gas, treating the atoms as fixed scatterers of densities $n_0$ and $n_1$.

(c) Find the equilibrium density $f_{eq}$ for the photons of the above frequency.

(d) According to Planck’s law, the density of photons at a temperature $T$ depends on their frequency $\omega$ as $f_{eq} = [\exp (\hbar\omega/k_B T) - 1]^{-1}/h^3$. What does this imply about the above cross sections?

(e) Consider a situation in which light shines along the $x$ axis on a collection of atoms whose boundary coincides with the $x = 0$ plane, as illustrated in the figure.

Clearly, $f$ will depend on $x$ (and $p_x$), but will be independent of $y$ and $z$. Adapt the Boltzmann equation you propose in part (b) to the case of a uniform incoming flux of photons with momentum $\vec{p} = \hbar\omega\hat{x}/c$. What is the penetration length across which the incoming flux decays?

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8. *Effusion*: The probability distribution for speed $c$ of particles of mass $m$ in a gas at temperature $T$ is proportional to $c^2 e^{-c^2/2\sigma^2}$, with $\sigma^2 = k_B T/m$. Some particles are allowed to leak (effuse) out of a small hole with diameter much less than the mean free path.
(a) Show that the probability distribution for speed of the escaping particles is proportional to $c^3 e^{-c^2/2\sigma^2}$.

(b) Find the average kinetic energy of the escaping particles.

(c) What is the fraction of escaping particles with kinetic energy greater than $E$?

† Reviewing the problems and solutions provided on the course web-page for preparation for Test 2 should help you with the above problems.