Ising spins on a hyperbolic lattice

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Introduction

The spatial dimension plays a large role in determining the behavior of magnetic and spin systems; one of the key components in determining a system's critical universality class is its spatial dimension. The literature shows ϵ-expansions around dimensions four and two, which follow from perturbative RG expansions of the Landau-Ginzburg hamiltonian and the XY-model, respectively. However, we are most interested in dimension three, which is of great physical relevance. Two-dimensional spin systems are special in many ways — they are the highest-dimensional Ising system that has been solved exactly, the two-dimensional XY model has algebraic decay instead of exponential, etc.. The dimensionality of four is also a interesting boundary state, at the limit of where the Gaussian solution to the Landau-Ginzburg Hamiltonian becomes correct. There are many ways to explain this, in particular by analyzing the relevance of perturbation corrections, and the high-temperature expansion in terms of self-avoiding loops. These loops can be represented in terms of random walks (without backtracking), and we can interpret the Gaussian Landau-Ginzburg solution as assuming that the random walks are independent. Since the random walk has a characteristic dimesion of two, in spatial dimensions four and higher, two random walks intersect each other almost never, and are, in fact, well-represented as independent objects [1].

These results assume Euclidean space; we are curious what happens when we add spatial curvature to the system. The case of positive curvature is less interesting, as the resultant *finite* system cannot exhibit a divergent correlation length, etc.. We thus consider spaces of (constant) negative curvature; we refer interchangably to a space with negative curvature and a hyperbolic space. In particular, we attempt to simulate $\mathbb{H}^2$, the hyperbolic plane. One of the most interesting features of the hyperbolic geometry is the behaviour of geodesics (the generalization of "straight lines"). In ordinary flat (Euclidean) space, given a line and disjoint point, there is a unique parallel line through that point, which is constructed through an auxiliary line perpendicular to both. These parallel lines never intersect, and are a constant distance from each other. In hyperbolic space, the sense of a parallel line is not quite as clear — though there is still a unique line with the shared common perpendicular, there are also infinitely many asymptotic lines through that point that also do not intersect the original line. Furthermore, the distance between the lines is not constant.

In terms of a statistical field theory, the transition to curved space appears as a new metric in the spatial integrals. However, for a simple two-state (Ising) spin system, it is much more convenient for us to use a discretized model. All that is necessary is a regular tiling of $\mathbb{H}^2$, and there are in fact an infinite number of such tilings, unlike in Euclidean space. This is because the sum of the angles of a triangle are only constrained to sum to *less* than 180°, whereas for Euclidean space the constraint is the much stricter equality condition.

Before we jump into calculations and simulations, we can think a bit about how adding spatial curvature might affect existing results. We recall that the high-temperature expansion of the partition function of the Landau-Ginzburg Hamiltonian is interpreted in terms of random walks; if we now consider these random walks in a manifold of negative curvature, the same non-backtracking random walks can behave very differently, since these manifolds have unbounded isoperimetric dimension — the relevance of the characteristic dimension of two is less clear. In the hyperbolic space, the perimeter of a graph grows exponentially in the diameter (in Euclidean space, this growth is *linear*), and it is not entirely clear how this "extra space" affects the dynamics of random walks and their avoidance properties, let alone the dimensional behavior of a Landau-Ginzburg system.

We also note that there are systems (i.e. the XY model) where topological defects have a significant impact on the overall energy; a lattice on a hyperbolic manifold can be thought of as a lattice where *every* site is a topological defect. Consideration of an XY model on a negative-curvature manifold thus presents an interesting area for future study.

Methods

In order to probe the critical properties of the Ising model, we use a Monte Carlo method to sample the spin states of a particular (finite) system. Our Hamiltonian is

$$\mathcal{H} = -J \sum_{\langle i,j \rangle} \sigma_i \sigma_j$$

where the spins σ_i take on the values ± 1. We adopt the convention of setting the dimensionless coupling constant $K = \frac{J}{k_B T}$. We note that since the sum is only over nearest-neighbor pairs, we can assign a "weight" to each bond of our lattice, as $W = \sigma_i \sigma_j$, that also takes on the values ± 1. Having the spins and weight (and thus magnetization and energy) be integer values allows for faster simulations, as floating-point arithmetic is only invoked to evaluate the Boltzmann factors. Notably, there are no rounding errors to cascade over the course of the simulation [3].

The particular structure of the hyperbolic plane prevents many of our normal techniques from being applied. Since the distance between "parallel" lines is not constant, we cannot create a sublattice with the same shapes — in a pentagonal lattice, we cannot simply combine N pentagons into a single, larger, pentagon, because of the divergence of the geodesics. This prevents the application of the Renormalization Group since the renormalization procedure is different at each iteration — there are no self-consistent recursion relations. Furthermore, it prevents us from choosing as our system a subset of the infinite graph plane that can tile the graph. As such, any simulation on a hyperbolic lattice cannot use periodic boundary conditions, so neutral boundary conditions (spins and bond weights of zero along the perimeter) must be used. Thus, we are simulating a truly finite system. For consistency, we use the same boundary conditions for the hyperbolic tiling and a reference tiling of a flat (Euclidean) plane. These boundary conditions become more and more important as the size of the lattice increases, since the perimeter grows faster than the diameter. When a sizeable portion of the spins are not experiencing the bulk environment, the magnetization estimate is likely to be off. This error can probably be corrected for by excluding the spins on the boundary from the calculation of the magnetization, but this has not yet been implemented.

Another challenge in tiling a hyperbolic plane is that there is no concise (implicit) form for describing the system — whereas spins on a square lattice can be described by a set of ones and zeros in a matrix, there is no convenient way to implicitly index the spins on a hyperbolic lattice. Thus, we must directly store the graph structure of our lattice with edges, vertices, and polygons, and the explicit topology that they comprise. We still have constant-time updates for spin flips, though, since the Hamiltonian is strictly local. The magnetization is just $m = \sum_i \sigma_i$, so changing a single spin changes m by two; the weight (proportional to energy) is a sum over bonds, but for a single spin flip, there are at most four affected bonds, and the energy changes by twice the energy of the bond. These local changes only require a constant amount of time, so each (attempted) spin flip has the same cost, independent of lattice size.

The hyperbolic lattice also suffers from the difficulty that it does not have a simple linear dimension — our sample system is not the same shape as one of the fundamental building blocks, and has no single length that describes it (the diameter being somewhat difficult to compute). This is in contrast to flat space, where it is easy to say that we have a $L \times L$ square lattice (which is in fact what our reference system is for this work). We thus can only refer to the size of our lattice in terms of the number of spins. For this work, simulations were performed on systems with 1600 and 10,000 spins.

No special (pseudo-)random-number generator was used; entropy was harvested from the FreeBSD `random()` system call, which uses a non-linear additive feedback random number generator with a large (approximately $16 \times ((2^{31}) - 1)$) period. Simulations were run on an Intel Core2 Quad (Q6600) CPU and ran for 10^5 to 10^7 MCS, depending on the run. At each step, an index was chosen at random into a global table of spins, and that spin was considered for flipping using a Metropolis algorithm. We consider a full Monte Carlo Step (MCS) to be attempting as many spin flips as there are spins in the lattice; we do not guarantee that each spin is updated on each MCS, because at each step we select a random spin to consider. Since the global table of spins was filled by concentrically enlarging the graph around its perimeter, there is no uniform correlation between spins separated by some distance in the table, and we do not expect that correlations in the random-number generator would cause systematic error, though this possibility was not explicitly checked by re-seeding or doing a comparison simulation using cryptographic-quality entropy.

For our simulations, we chose to use a square lattice for the flat (Euclidean) reference system, and a lattice of right-angled pentagons for the curved (hyperbolic) system (the angles sum to more than 2π because when we represent $\mathbb{H}^2$ in a Euclidean plane, geodesics are curved lines). In order to study these systems, we created a general framework to create a convex graph of spins, taking as parameters the valence at each spin and the type of polygon to use for the tiling. Starting with a single polygon, the graph was enlarged by proceeding around the perimeter and adding a polygon at a time.

During the Monte Carlo simulations, energy and magnetization pairs were stored, and measurements were made at intervals of 25 full MCS; additionally, the first 10^5 MCS were discarded as equilibration from the uniform-spin initial condition. Though random spin initial conditions were implemented, uniform-spin was preferred for data collection, because we could place some bounds on the equilibration time of the system. A run at a single temperature took about 50 minutes to complete.

Results and Discussion

There is little to discuss, as we have not been able to acquire reliable data. A set of trials was performed at some 40 K values between 0.221400 and 0.222000 (note that 0.221654 is a good estimate of the critical coupling from other Monte Carlo simulations [2]), using a lattice size of 10^4 spins, and undertaking 10^5 MCS for each coupling. When the expected magnetization and susceptibility were computed, though, the magnetization was far from zero for almost all data points, and was inconsistent with a discontinuous phase transition.

We concluded that we had not sampled the system for long enough, and thus decided to try a smaller lattice of 1600 spins, and to evaluate 10^7 MCS. Though the noise in this data set was greatly reduced, we still could not ascertain any critical behavior in either the Euclidean or hyperbolic data sets. In retrospect, it is possible that our entropy pool was not large enough — the quoted period of $16 \times (2^{31} - 1)$ is about 10^{11}, and our latter run sampled 1.6×10^{10} spin configurations. Unfortunately, these calculations are too expensive for another set to be completed before the submission deadline, so we are unable to confirm this hypothesis at this time.

Future work

We would like to confirm or deny that our current lack of data indicating critical behavior arises from insufficient entropy in the sampling pool; this can be easily done by replacing the `random()` system call with a cryptographically-secure variant.

Since our framework for constructing a lattice is quite general and easily re-parametrized, it would be interesting to compare results for the (all Euclidean) hexagonal, triangular, and square lattices, as well as to see if the parameters of the hyperbolic tiling led to different critical behavior (i.e. if there was a dependence on the magnitude of the negative curvature), but such investigations are beyond the scope of the present work.

Another key concept in critical phenomena is the particular scaling exponents of a universality class. It is possible to compute such exponents from Monte Carlo simulations, for example using the histogram method, but all of these methods depend on the length-dependence of various properties of the system. To be more correct we should compute a finite-size correction to our results [2]. For our purposes, this requires running simulations on lattices of different sizes, a computation (and data analysis) that is beyond the scope of the present work.

Bibliography

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