Detecting Nonstationary Scaling of Diffusion Processes with Generalized Entropies

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Scafetta et. al. introduced in 2001-2002 their Diffusion Entropy Analysis (DEA) for calculating the temporal scaling exponent of diffusion processes. A generalization involving Tsallis entropy allows one to extract weakly nonstationary scaling exponents, and consequently, characterize the degree of departure from stationarity. Here, we establish that this approach is special as it implicitly makes use of all the moments of the probability density function which describes the diffusion process – unlike other entropy functionals under consideration. By making use of recent observations by Frank et. al. that the Sharma-Mittal entropy include "Gaussian", Rényi and Tsallis entropies as special cases, we show that the interpretive power of DEA can be increased if the Sharma-Mittal entropy is adopted. As the DEA is the only method which detects scaling impartial to both Gaussian and Lévy statistics, our extension provides yet another complementary interpretation for the nonextensive parameter q ubiquitous in nonextensive statistical mechanics.

I. INTRODUCTION

A time series is an ordered sequence of numbers. They are frequently encountered whenever data is recorded sequentially. If the time series consists of a sequence of trajectories, one can define a mapping of the time series to an (estimated) underlying probability density function (PDF) by considering the distribution of all possible contiguous trajectories. In this sense, a given time series can be seen as a generator of a diffusion process. By comparing the length of these trajectories in time and the associated PDF generated by these trajectories, one can recover a temporal scaling exponent which can be used to characterize different universality classes.

Scafetta et. al. have devised a Diffusion Entropy Analysis (DEA) method [1] for extracting the temporal scaling exponent of diffusion processes valid for both Gaussian statistics (which have finite variance) *and* Lévy statistics (which do not). Respective examples include fractal Brownian motion, and Lévy flights. The DEA rests on calculating the rate of entropy production as a function of trajectory lengths. A number of applications papers have been published [2–4] which show that while "traditional" methods based on the scaling of variance alone correctly identify scaling exponents generated from Gaussian statistics, they fail for Lévy statistics. This is a consequence of the possibility that the entropy of a probability distribution can be finite, even when its variance is not (e.g. a Cauchy distribution). The authors emphasize [5], that if these traditional methods are used *in conjunction* with DEA, one can better characterize scaling properties. Some examples of DEA applications have since appeared [6–9] which reflect this confluence.

It should be noted that DEA relies on the assumption that the diffusion process is stationary, that is, the statistics of the PDF are not changing with time. Stationarity is important because there are detectable consequences when one departs towards nonstationarity. Scafetta et al. consider a generalized DEA with the goal of handling weak nonstationarities in the scaling exponent [10, 11]. The crucial observation made is that DEA relies on estimating the Boltzmann-Gibbs entropy, so that a reasonable generalization is to replace the classical entropy functional with another that parameterizes a broader class of entropic measures (of which the Boltzmann-Gibbs entropy is one special case). In this way, Scafetta et. al. have shown that DEA based on Tsallis entropy achieves their objective. Consequently, one can associate the degree of nonextensivity in the entropy with the degree of departure from stationarity (and/or ordinary statistics) towards nonstationarity (and/or anomalous statistics).

This "degree of nonextensivity" is of course, the so-called "entropic measure" q that is ubiquitous in nonextensive statistical mechanics inspired by Tsallis' original paper in 1988 [12], and its enthusiastic characterization and application by researchers since that time [13–15]. Given the promise of nonextensive statistical mechanics as a possible generalization of ordinary statistical mechanics, it is helpful to examine whether the scaling exponents uncovered with generalized DEA (which uses Tsallis entropy) can suggest other relationships between novel and ordinary statistical mechanics. The time dependence of one dimensional diffusion processes serves as a good model system for such an analysis, in part, because of the copious examples of one-dimensional trajectories which do *not* follow ordinary diffusion. This effort is directed towards establishing yet another interpretation of the "nonextensive parameter" q ubiquitous in the formalism of nonextensive statistical mechanics.

In Section II we introduce the notion of scaling of diffusion processes, and characterize the temporal scaling behaviour of ordinary diffusion. In Section III the properties of weak nonstationary scaling investigated by Scafetta et. al. [10] is introduced. We elaborate on their first order derivation, and show that the Tsallis entropy can be substituted with a so-called "Gaussian entropy" which also results in the same set of equations. We also show explicitly that the Rényi entropy cannot aid in the

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determination of scaling exponents. In Section IV we interpret the results in obtained in Section III in the context of Sharma-Mittal entropy. The inspiration for this analysis originated from a series of observations made by Frank. et al. [16–18], in particular, the observation that the Sharma-Mittal entropy [19] precedes the Tsallis entropy and additionally includes it as a special case. Finally, in Section V we provide a summary and outlook for the next steps of this theory. The Appendices contain derivations for selected parts of the analysis.

II. SCALING AND ORDINARY DIFFUSION

A one-dimensional diffusion process can be characterized by a probability density function (PDF) $p = p(x,t)$. This PDF describes the spatial distribution of an ensemble of particles which have undergone diffusion from the same point at any time $t \geq 0$. If we assume that p is scale-invariant, then it should satisfy the homogeneity condition

$$bp(x,t) = p(b^{y_x}x, b^{y_t}t) \tag{1}$$

for any given constant b, and fixed (time independent) scaling exponents y_x and y_t. In Appendix A, it is shown that with the constraint that probability density must be normalized to unity, the functional form of p must be of the form (to make contact with Eq. 2 of [1])

$$p(x,t) = \frac{1}{t^\delta} F\left(\frac{x}{t^\delta}\right), \delta = \frac{1}{y^t} \tag{2}$$

in which F is also a normalized probability density. When δ is independent of t, a stationary diffusion process is described by p. In order to extract information about δ, we follow [1] by considering the classical entropy functional

$$S[p] = -\int_{-\infty}^{\infty} \mathrm{d}x\, p \ln p. \tag{3}$$

Substituting Eq. 2 into Eq. 3 and measuring time t in units of $\tau = \ln t$, this results in the linear equation

$$S(\tau) = -\langle \ln F \rangle + \delta\tau \equiv A + \delta\tau \tag{4}$$

whose derivation can be found in Appendix B. Here, A is a constant, and is independent of τ. By taking a derivative with respect to τ, it follows that $dS/d\tau = \delta$. Hence, if the quantity $S(\tau)$ can be obtained, δ can be obtained by considering its slope $dS/d\tau$ on an "S-τ plot". This is true even if p or F in Eq. 2 are not known analytically.

A. Example: Classical Diffusion

The partial differential equation which governs classical diffusion in one dimension is

$$\frac{\partial p(x,t)}{\partial t} = \frac{\sigma^2}{2} \frac{\partial^2 p(x,t)}{\partial x^2} \tag{5}$$

and has the solution

$$p(x,t) = \frac{\exp\left(-\frac{x^2}{2\sigma^2 t}\right)}{\sqrt{2\pi\sigma^2 t}} \propto \frac{1}{\sqrt{t}} \cdot \frac{\exp\left(-\frac{1}{2\sigma^2}\left(\frac{x}{\sqrt{t}}\right)^2\right)}{\sqrt{2\pi\sigma^2}} \tag{6}$$

for a point source initial condition. By comparing with Eq. 2, the scaling exponent for classical diffusion is $\delta = \frac{1}{2}$. Equation 4 can also be evaluated for this particular probability density to give

$$S(\tau) = \frac{1}{2}\ln(2\pi e\sigma^2) + \frac{\tau}{2} \tag{7}$$

which confirms that $\delta = \frac{1}{2}$ since it is the coefficient of τ in Eq. 7. The derivation of this is given in Appendix C.

Presumably, if a diffusion process does not obey the classical diffusion equation, $\delta \neq \frac{1}{2}$ would be suggestive of this. However, let us consider the case in which δ is not constant and has a weak dependence on τ. In this situation, the scaling is time dependent and so, nonstationary. This is considered in the next section.

III. NONSTATIONARY SCALING

Here, we present the salient results of Scafetta et. al. [10] in detail. The objective is to derive a set of equations which permit the detection of nonstationary scaling. Consider the case where the scaling δ is not a constant. Then

$$p(x,t) = \frac{1}{t^{\delta(t)}} F\left(\frac{x}{t^{\delta(t)}}\right) \tag{8}$$

and so $S(t) = \delta(t)\ln t + A$. This is simply Eq. 4 with $\delta \to \delta(t)$. Suppose further that

$$\delta(t) = \delta_0 + \delta_1 \ln t + \mathcal{O}(\ln^2 t). \tag{9}$$

Then, we can define $\delta(\tau) = \delta_0 + \delta_1\tau + \mathcal{O}(\tau^2)$ so that the entropy with weak nonstationarity reads

$$S(\tau) = A + \delta_0\tau + \delta_1\tau^2 + \mathcal{O}(\tau^3). \tag{10}$$

Due to the presence of the quadratic term $\delta_1\tau^2$, the scaling exponent is no longer characterized by a linear increase in entropy with respect to τ. In order to suppress this quadratic term, one approach is to introduce an additional parameter which can be varied. This can be achieved by using the Tsallis entropy.

Since diffusion occurs in a dissipative system, the scaling exponent cannot exceed the situation which corresponds to unhindered diffusion. This "ballistic" regime imposes an upper bound of $\delta(t) \leq 1$ for all t.

A. Detection via Tsallis Entropy

Here, we show how the the Tsallis entropy [12] can be used to determine whether $\delta_1 = 0$ or otherwise. The

Tsallis entropy is defined as

$$S_q^T(t) = \frac{1}{q-1}\cdot\left(1-\int_{-\infty}^{\infty}\mathrm{d}x\, p^q(x,t)\right) \quad (11)$$

and in the limit as $q \to 1$ we recover the normal entropy. The proof is provided in Appendix D. Substituting $\epsilon = q-1$ and assuming that $\epsilon \ll 1$we get to first order in ϵ

$$S_{1+\epsilon}^T(t) = -\langle \ln p\rangle - \frac{\epsilon}{2}\langle \ln^2 p\rangle + \mathcal{O}(\epsilon^2) \quad (12)$$

$$= -\int_{-\infty}^{\infty}\mathrm{d}x\, p\left(\ln p + \frac{\epsilon}{2}\ln^2 p\right) \quad (13)$$

Now, consider substituting Eq. 8 and Eq. 9 into Eq. 13. This is shown in Appendix E. The result to first order in ϵ is

$$S_{1+\epsilon}^T(\tau) = \delta\tau - \langle \ln F\rangle - \frac{\epsilon}{2}\left[\langle \ln^2 F\rangle - 2\delta\tau\langle \ln F\rangle + \delta^2\tau^2\right]. \quad (14)$$

Since $\delta = \delta(\tau) = \delta_0 + \delta_1\tau + \mathcal{O}(\tau^2)$, to second order in τ, Eq. 14 reads

$$\begin{aligned} S_{1+\epsilon}^T(\tau) = &-\left[\langle \ln F\rangle + \frac{\epsilon}{2}\langle \ln^2 F\rangle\right] \\ &+\tau\left[\delta_0\left(1+\epsilon\langle \ln F\rangle\right)\right] \\ &-\tau^2\left[\frac{\epsilon}{2}\delta_0^2 - \delta_1\left(1+\epsilon\langle \ln F\rangle\right)\right] \end{aligned} \quad (15)$$

This equation is interesting from at least four perspectives:

- The zeroth order term $-\langle \ln F\rangle - \frac{\epsilon}{2}\langle \ln^2 F\rangle$ is a first order perturbation on the zeroth order term obtained in Eq. 4.
- The first order term $\delta_0(1+\epsilon\langle \ln F\rangle)$ is also a first order perturbation of the result obtained in Eq. 4. However, the perturbation coefficient is $\langle \ln F\rangle$ as opposed to $\langle \ln^2 F\rangle/2$.
- The entropy increases *linearly* with respect to τ (just like ordinary DEA) when the second order term vanishes. This occurs for the special value of

$$\epsilon = \frac{\delta_1}{\delta_0^2/2 - \delta_1\langle \ln F\rangle} = \frac{\delta_1}{\delta_0^2/2 + A\delta_1} \quad (16)$$

  which recovers the result in [10].

- Once ϵ has been found, Eq. 16 can be inverted to solve for δ_1

$$\delta_1 = \frac{1}{2}\frac{\epsilon\delta_0^2}{1+\epsilon\langle \ln F\rangle} = \frac{1}{2}\frac{\epsilon\delta_0^2}{1-\epsilon A} \quad (17)$$

  which characterizes the degree of nonstationarity in the scaling exponent. This result is also in accord with [10].

Equation 17 shows that $\epsilon = 0$ necessarily implies that $\delta_1 = 0$ which holds only when the diffusion process is stationary. Observe that this is a particularly sensitive measure of nonstationarity, since

$$\frac{1}{\epsilon}\frac{d\delta_1}{d\epsilon} = \frac{\delta_0}{2}\cdot\frac{1}{\epsilon}\cdot\frac{1}{\left(1+\epsilon\langle \ln F\rangle\right)^2} \quad (18)$$

is large at the vicinity of $\epsilon \sim 0$. The squared term is also large whenever $\epsilon > 0$ since the quantity $\langle \ln F\rangle < 0$. This explains the sensitivity of this approach for weak nonstationarities [10].

B. Detection via Gaussian Entropy

Frank [16] defines the "Gaussian entropy" as

$$S_g^G(t) = \frac{1-\exp\left[(g-1)\int_{-\infty}^{\infty}\mathrm{d}x\, p(x,t)\ln p(x,t)\right]}{g-1}. \quad (19)$$

We will elaborate on this peculiar form in Section IV. For the moment, if we set $\epsilon = g-1$ then

$$S_{1+\epsilon}^G(t) = \frac{1}{\epsilon}\left(1-e^{\epsilon\langle \ln p\rangle}\right) = -\sum_{n=1}^{\infty}\frac{\epsilon^{n-1}\langle \ln p\rangle^n}{n!}. \quad (20)$$

Compared with the associated series for the Tsallis entropy (cf. Eq. E8)

$$S_{1+\epsilon}^T = -\sum_{n=1}^{\infty}\frac{\epsilon^{n-1}\langle \ln^n p\rangle}{n!} \quad (21)$$

these two are identical if one makes a correspondence between $\langle \ln p\rangle^n$ and $\langle \ln^n p\rangle$. Hence, analogous to Eq. 15 we trivially obtain

$$\begin{aligned} S_{1+\epsilon}^G(\tau) = &-\left[\langle \ln F\rangle + \frac{\epsilon}{2}\langle \ln F\rangle^2\right] \\ &+\tau\left[\delta_0\left(1+\epsilon\langle \ln F\rangle\right)\right] \\ &-\tau^2\left[\frac{\epsilon}{2}\delta_0^2 - \delta_1\left(1+\epsilon\langle \ln F\rangle\right)\right] \end{aligned} \quad (22)$$

and Eq. 16 and Eq. 17 follow. Within this correspondence is a subtle property which distinguishes the Tsallis entropy from the Gaussian entropy. Although both entropies admit the same equations which define the nonstationary condition, the Gaussian entropy only requires of *powers* of $\langle \ln p\rangle$ whereas the Tsallis entropy requires involves *moments* of $\langle \ln p\rangle$. Hence, there is a computational advantage to using the Gaussian entropy over the Tsallis entropy – the quantity $\langle \ln p\rangle$ need only be computed once, after which all the higher order powers can be easily obtained. The moments of $\langle \ln p\rangle$ cannot be obtained in this manner. Furthermore, the Gaussian entropy functional is simpler to evaluate as the parameter g is not found inside an integral.

Apart from this computational difference, the distinguishing factor between the Tsallis and Gaussian approach seems to lie not in the first and higher order expansions, but rather, the zeroth order term. Since the Tsallis entropy extracts information from the *entire* probability distribution, it should be possible to use the *intercept* of an S-τ plot in order to refine the estimate for the entropy. This observations is elaborated in Section IV in which the Sharma-Mittal entropy is discussed.

C. Failure of Detection via Rényi Entropy

From analysing the results obtained for the DEA approach using Tsallis and Gaussian entropy, one can infer that a criterion for detecting nonstationary scaling is that the entropy functional should modify perturbatively the coefficients of the entropy when expanded in powers of τ. In this section, we show that the Rényi entropy [20] lacks this property, and hence, is not useful for detecting nonstationarities by itself. The Rényi entropy is given by

$$S_r^R(t) = \frac{1}{1-r} \ln \left(\int_{-\infty}^{\infty} \mathrm{d}x \, p^r(x,t) \right) \tag{23}$$

which in the limit $r \to 1$ also reduces to the Boltzmann-Gibbs entropy. Consider defining $\epsilon = 1 - r$ so that

$$S_{1-\epsilon}^R(t) = \frac{1}{\epsilon} \ln \left(\int_{-\infty}^{\infty} \mathrm{d}x \, p^{1-\epsilon} \right). \tag{24}$$

Substituting Eq. 2 into Eq. 24 gives

$$\int_{-\infty}^{\infty} \mathrm{d}x \, p \cdot p^{-\epsilon} = \int_{-\infty}^{\infty} \mathrm{d}x \, \frac{1}{t^\delta} F\left(\frac{x}{t^\delta}\right) \left[t^{\delta\epsilon} F^{-\epsilon} \left(\frac{x}{t^\delta}\right)\right] \tag{25}$$

$$= t^{\delta\epsilon} \int_{-\infty}^{\infty} \mathrm{d}y \, F(y) \left[F^{-\epsilon}(y)\right] \tag{26}$$

$$= t^{\delta\epsilon} \langle F^{-\epsilon} \rangle. \tag{27}$$

Hence,

$$S_{1-\epsilon}^R(t) = \delta \ln t + \frac{1}{\epsilon} \ln \langle F^{-\epsilon} \rangle. \tag{28}$$

However, this shows that ϵ does not perturbatively modify the $\delta \ln t$ term, so is consistent with observations that the Rényi entropy only displaces the S-τ plot vertically and does not modify its curvature [21].

IV. SHARMA-MITTAL ENTROPY

The Sharma-Mittal entropy [16, 17, 19] is a two parameter entropy defined by

$$S_{q,r}^{SM}(t) = \frac{1 - \left(\int_{-\infty}^{\infty} \mathrm{d}x \, p(x,t)^r \right)^{\frac{q-1}{r-1}}}{q-1}. \tag{29}$$

This entropy contains a number of interesting special cases. If $q = r$, the Tsallis entropy is recovered. If $r = 1$ and $q \neq 1$, the Gaussian entropy is recovered. If $q = 1$ and $r \neq 1$, the Rényi entropy is recovered. Finally, if both $q = r = 1$, the Boltzmann-Gibbs entropy is recovered. If this Sharma-Mittal entropy is rewritten as (c.f. Eq. 6.132 of [16])

$$S_{q,r}^{SM}(t) = \frac{1 - \exp\left[(1-q) \, S_r^R\right]}{q-1} \tag{30}$$

then this is identical to the functional form of Eq. 19, except the Rényi entropy S_r^R replaces the normal Boltzmann-Gibbs entropy. This means that the zeroth order term in the expansion of $S_{q,r}^{SM}(t)$ about $q = 1$ is no longer a function of the powers of the Boltzmann-Gibbs entropy, but rather, powers of the Rényi entropy. Therefore, the parameter r in the Sharma-Mittal entropy becomes an additional variable which can be modified so that the entropy at time $\tau = 0$ matches the expected entropy for the underlying distribution. For example, recall that the normal probability distribution maximizes the entropy for fixed variance. If the underlying distribution were otherwise, the entropy would be less than this maximum entropy. The parameter r can be adjusted so that the modified entropies coincide. If we define the metric $\sqrt{q^2 + r^2}$ as a means for determining how far from stationarity a system is, in a sense, adjusting both q *and* r this allows one to extract the *minimum* "distance" one must travel in the Sharma-Mittal parameter space in order to recover ordinary statistics. This is important since the Sharma-Mittal entropy satisfies a number of unique positivity and closure relationships [17].

V. FINAL REMARKS

As a followup to the ideas presented above, here are a few items which would be desirable to explore. They are as follows:

- Perform an numerical experiment to verify that the DEA can be built from the Gaussian entropy rather than the Tsallis entropy;

- Extend the DEA implementation to the Sharma-Mittal case, and evaluate whether the parameters q and r obtained from numerical experiments coincide with the theory above; and

- Investigate second order nonstationarities in the scaling exponent, i.e. $\delta(\tau) = \delta_0 + \delta_1 \tau + \delta_2 \tau^2$, and determine whether δ_2 can be characterized by a quadratic increase in DEA (preliminary results seem to suggest that the results to second order modify the first order results in a non-perturbative manner).

Appendix A: HOMOGENEITY CONSTRAINT

If a probability density function $p(x,t)$ is homogeneous, then it must satisfy Eq. 1, namely $bp(x,t) = p(b^{y_x}x, b^{y_t}t)$ for constants b, y_x and y_t. This means that

$$p(x,t) = \frac{1}{b}p(b^{y_x}x, b^{y_t}t). \quad \text{(A1)}$$

To eliminate the direct functional dependence of p on t, choose $b^{y_t}t = 1$. Then

$$p(x,t) = \frac{1}{t^{1/y^t}}p(t^{-y_x/y_t}x, 1) \equiv \frac{1}{t^{1/y^t}}F\left(\frac{x}{t^{y_x/y_t}}\right). \quad \text{(A2)}$$

The crucial observation is that F is an unknown probability density function. This permits the simplification of Eq. A4. At any given time t, the probability density p should remain normalized. Hence

$$\int_{-\infty}^{\infty} \frac{1}{t^{1/y^t}}F\left(\frac{x}{t^{y_x/y_t}}\right)\,\mathrm{d}x = 1. \quad \text{(A3)}$$

Now, consider a change of variables $\rho = xt^{-y_x/y_t}$. Then the integral becomes

$$t^{(y_x-1)/y_t}\int_{-\infty}^{\infty} F(\rho)\,\mathrm{d}\rho = 1 \quad \text{(A4)}$$

$$t^{(y_x-1)/y_t} = 1 \quad \text{(A5)}$$

which can only hold for all t provided that $y_x = 1$. This implies that

$$p(x,t) = \frac{1}{t^{1/y^t}}F\left(\frac{x}{t^{1/y_t}}\right)$$

and setting $\delta = 1/y^t$ the result in [1] is recovered. We can verify that this functional form leads to a normalized probability density with the change of variables $x = yt^{\delta}$ as

$$\int_{-\infty}^{\infty}\mathrm{d}x\,p(x,t) = \int_{-\infty}^{\infty}\frac{\mathrm{d}x}{t^{\delta}}F\left(\frac{x}{t^{\delta}}\right) = \int_{-\infty}^{\infty}\mathrm{d}y\,F(y) = 1$$

which holds for all time t.

Appendix B: SCALING EXPONENT AND ENTROPY

In this section, the scaling exponent δ of a probability density function p is related to the classical entropy S given in Eq. 3. By considering the natural logarithm of Eq. 2, we obtain

$$\ln p = \ln F\left(\frac{x}{t^{\delta}}\right) - \delta\ln t. \quad \text{(B1)}$$

Substituting Eq. B1 into Eq. 3 gives

$$S(t) = -\int_{-\infty}^{\infty}\mathrm{d}x\,\frac{1}{t^{\delta}}F\left(\frac{x}{t^{\delta}}\right)\left[\ln F\left(\frac{x}{t^{\delta}}\right) - \delta\ln t\right] \quad \text{(B2)}$$

$$= -\int_{-\infty}^{\infty}\mathrm{d}x\,\frac{1}{t^{\delta}}F\left(\frac{x}{t^{\delta}}\right)\ln F\left(\frac{x}{t^{\delta}}\right) + \delta\ln t \quad \text{(B3)}$$

$$= A + \delta\tau \quad \text{(B4)}$$

with $\tau = \ln t$. Equation B4 is the same as Eq. 4 in Section I. The quantity A is defined as

$$A \equiv -\int_{-\infty}^{\infty}\mathrm{d}x\,\frac{1}{t^{\delta}}F\left(\frac{x}{t^{\delta}}\right)\ln F\left(\frac{x}{t^{\delta}}\right) \quad \text{(B5)}$$

and with the substitution $y = x/t^{\delta}$ it simplifies to

$$A = -\int_{-\infty}^{\infty}\mathrm{d}y\,F(y)\ln F(y) = -\langle\ln F\rangle \quad \text{(B6)}$$

which is independent of t (and of $\tau = \ln t$). Interestingly we find that quantity A is precisely the classical entropy of the probability density function F. Hence, on an "S-τ plot" of $S(\tau) = A + \delta\tau$, the intercept with the S axis is equal to A.

Appendix C: ENTROPY OF A GAUSSIAN

The entropy of a Gaussian probability density given by Eq. 6 can be computed via straightforward integration. Since

$$\ln p(x,t) = -\frac{1}{2}\ln(2\pi\sigma^2 t) - \frac{x^2}{2\sigma^2 t} \quad \text{(C1)}$$

then

$$-\int_{-\infty}^{\infty} p(x,t)\ln p(x,t)\,\mathrm{d}x = \frac{1}{2}\ln(2\pi\sigma^2 t) + \frac{\langle x^2\rangle}{2\sigma^2 t} \quad \text{(C2)}$$

Since the second moment of p is $\langle x^2\rangle = \sigma^2 t$, the entropy is $S(t) = \frac{1}{2}\ln(2\pi\sigma^2 et)$, or simply

$$S(\tau) = \frac{1}{2}\ln(2\pi\sigma^2 e) + \frac{\tau}{2} \quad \text{(C3)}$$

which recovers the result in Eq. 7 when time units of $\tau = \ln t$ are used.

Appendix D: TSALLIS LIMITING BEHAVIOUR

In this section we prove that as $q \to 1$, the Boltzmann-Gibbs entropy is recovered from the Tsallis entropy. Starting from the Tsallis entropy

$$S_q(t) = \frac{1}{q-1}\cdot\left(1 - \int_{-\infty}^{\infty}\mathrm{d}x\,p^q(x,t)\right) \quad \text{(D1)}$$

we can rewrite this as

$$S_q(t) = \frac{1 - \int_{-\infty}^{\infty}\mathrm{d}x\,e^{q\ln p}}{q-1}. \quad \text{(D2)}$$

When $q = 1$ both the numerator and denominator vanish, so we can apply l'Hôpital's rule to find that

$$\lim_{q\to 1} S_q(t) = \lim_{q\to 1} \frac{\frac{d}{dq}\left(1 - \int_{-\infty}^{\infty} \mathrm{d}x\, e^{q\ln p}\right)}{\frac{d}{dq}(q-1)} \tag{D3}$$

$$= -\lim_{q\to 1} \int_{-\infty}^{\infty} \mathrm{d}x\, e^{q\ln p} \ln p \tag{D4}$$

$$= -\int_{-\infty}^{\infty} \mathrm{d}x\, p \ln p \tag{D5}$$

which recovers the usual entropy.

Appendix E: TSALLIS ENTROPY EXPANSIONS

The Tsallis entropy is given by

$$S_q(t) = \frac{1}{q-1} \cdot \left(1 - \int_{-\infty}^{\infty} \mathrm{d}x\, p^q(x,t)\right) \tag{E1}$$

and substituting $\epsilon = q - 1$ we get

$$S_{1+\epsilon}(t) = \frac{1}{\epsilon} \cdot \left(1 - \int_{-\infty}^{\infty} \mathrm{d}x\, p^{1+\epsilon}(x,t)\right) \tag{E2}$$

$$= \frac{1}{\epsilon} \cdot \left(1 - \int_{-\infty}^{\infty} \mathrm{d}x\, p \cdot e^{\epsilon \ln p}\right) \tag{E3}$$

$$= \frac{1}{\epsilon} \cdot \left(1 - \int_{-\infty}^{\infty} \mathrm{d}x\, p \cdot \left(\sum_{n=0}^{\infty} \frac{\epsilon^n \ln^n p}{n!}\right)\right) \tag{E4}$$

$$= \frac{1}{\epsilon} - \sum_{n=0}^{\infty} \int_{-\infty}^{\infty} \mathrm{d}x\, p \cdot \frac{\epsilon^{n-1} \ln^n p}{n!} \tag{E5}$$

$$= \frac{1}{\epsilon} - \frac{1}{\epsilon} - \sum_{n=1}^{\infty} \int_{-\infty}^{\infty} \mathrm{d}x\, p \cdot \frac{\epsilon^{n-1} \ln^n p}{n!} \tag{E6}$$

As a result, we obtain

$$S_{1+\epsilon}(t) = -\sum_{n=1}^{\infty} \int_{-\infty}^{\infty} \mathrm{d}x\, p \cdot \frac{\epsilon^{n-1} \ln^n p}{n!} \tag{E7}$$

$$= -\sum_{n=1}^{\infty} \frac{\epsilon^{n-1} \langle \ln^n p \rangle}{n!} \tag{E8}$$

Here, we use

$$\langle \ln^n p \rangle = \int_{-\infty}^{\infty} \mathrm{d}x\, p \ln^n p. \tag{E9}$$

Expanding to quadratic order for illustrative purposes, the Tsallis entropy reads

$$S_{1+\epsilon}(t) = -\langle \ln p \rangle - \frac{\epsilon}{2} \langle \ln^2 p \rangle - \frac{\epsilon^2}{6} \langle \ln^3 p \rangle - \mathcal{O}(\epsilon^3) \tag{E10}$$

if $\epsilon \ll 1$. This shows that the moments of $\ln p$ are intimately related to the expansion of the entropy about $\epsilon = 0$. The dependence on p can be integrated out if we substitute $\ln p = \ln F - \delta\tau$ (which is a consequence of Eq. 2) into equation Eq. E7 and make use of the fact that the probability density is normalized. In terms of $\tau = \ln t$ we have $S_{1+\epsilon}(\tau) = -\sum_{n=1}^{\infty} \int_{-\infty}^{\infty} \mathrm{d}x\, p \left[\epsilon^{n-1} (\ln F - \delta\tau)^n / n!\right]$. Using a binomial series expansion, this becomes

$$S_{1+\epsilon}(\tau) = \sum_{n=1}^{\infty} \frac{\epsilon^{n-1}}{n!} \sum_{m=0}^{n} \binom{n}{m} (-1)^{m+1} (\delta\tau)^m \langle \ln^{n-m} F \rangle \tag{E11}$$

which is in terms of the moments of $\ln F$ instead of $\ln p$. To first order in ϵ, this series yields

$$S_{1+\epsilon}(\tau) = \delta\tau - \langle \ln F \rangle - \frac{\epsilon}{2} \left[\langle \ln^2 F \rangle - 2\delta\tau \langle \ln F \rangle + \delta^2 \tau^2\right] \tag{E12}$$

which is Eq. 14.

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