

RG Analysis of the Kolmogorov Cascade in Isotropic Turbulence

David Benjamin*

Department of Physics, Harvard University

(Dated: May 12, 2008)

We review the application of renormalization group (RG) methods to the problem of isotropic turbulence, with attention to Kolmogorov's phenomenological scaling law. We see that RG reproduces the form of the 5/3 law and can also be used to obtain estimates of the Kolmogorov constant that agree with experiment and numerical simulation. The articles reviewed also confirm the universality of the scaling law and the constant. Finally, we posit a new form for correlations in Fourier space motivated by Kolmogorov's theory and use it to apply RG to a modified Navier-Stokes equation.

I. INTRODUCTION

More than 100 years since the Navier-Stokes equation of fluid mechanics were developed, turbulence remains one of the outstanding problems of classical physics. Very few rigorous analytic results exist, and in practical settings empirical rules of thumb predominate. The most successful theoretical result in the field is Kolmogorov's phenomenological picture of an "energy cascade" [1]. This so-called K41 theory states that the energy distribution in Fourier space of a turbulent fluid obeys the power law $E(k) = K_K \epsilon^{2/3} k^{-5/3}$, where ϵ is the rate of energy dissipation and K_K is a universal constant. Kolmogorov's rigorous derivation is complicated, but the result follows easily from a heuristic argument and dimensional analysis [2]. Since turbulence is the transfer of energy from small wavenumbers (laminar flow) to large wavenumbers (eddies), there exist upper and lower cut-off wavenumbers k_d and k_i such that energy only enters the system for $k < k_i$ and is only dissipated on scales $k > k_d$. Then we may assume that for $k_d \ll k \ll k_i$ the behavior is scale-invariant, and thus the energy spectrum must obey a power law. For ϵ defined as the rate at which energy is supplied and dissipated, we further assume that the transfer of energy between wavenumbers $k_d \ll k \ll k_i$ takes the form of a wavenumber-independent flux to higher wavenumbers. Then ϵ and k are the only dimensionful parameters, and the K41 law follows. Note that the result technically only applies to isotropic turbulence. In practice, however, anisotropies, such as the mean fluid flow in a pipe, occur on a large scale for $k > k_i$. Thus experimental verification of the K41 theory is possible. In general, it is confirmed up to a small correction (~ 0.04) to the exponent 5/3, with Kolmogorov's constant found to be $K_K = 1.6 \pm 0.1$ [3].

II. EARLY RESULTS

We can write the Navier-Stokes equation as [3]

$$(\partial_t + \nu k^2)u_\alpha(\mathbf{k}) = M_{\alpha\beta\gamma}(\mathbf{k}) \int d^3j u_\beta(\mathbf{j})u_\gamma(\mathbf{k} - \mathbf{j}) \quad (1)$$

where $M_{\alpha\beta\gamma} = (-i/2)(k_\beta D_{\alpha\gamma}(\mathbf{k}) + k_\gamma D_{\alpha\beta}(\mathbf{k}))$ and $D_{\alpha\beta} = \delta_{\alpha\beta} - k_\alpha k_\beta / k^2$. Here ν is the viscosity and $\mathbf{u}(\mathbf{k})$ is the Fourier transform of the fluctuation of the velocity field about its mean value. Fourier transforming in time as well yields

$$(i\omega + \nu k^2)u_\alpha(\mathbf{k}, \omega) = \lambda M_{\alpha\beta\gamma}(\mathbf{k}) \int \frac{d^3q}{(2\pi)^d} d\omega' u_\beta(\mathbf{q}, \omega') u_\gamma(\mathbf{k} - \mathbf{j}, \omega - \omega') \quad (2)$$

for $\lambda = 1$. This addition is non-trivial because λ and ν vary under RG. Forster, Nelson, and Stephen (FNS) added a stochastic force term $f_\alpha(\mathbf{k}, \omega)$ to the RHS of Equation 2 [4]. The purpose of the random force term is to input energy, so that there is a steady-state flow of energy from low to high wavenumbers. We expect the energy distribution $E(k)$ to be independent of the details of the random force as long as $f(\mathbf{k})$ vanishes for $k < k_i$. As in the case of the Langevin equation, calculations require the correlation function of the random force.

FNS initially applied RG to the Navier-Stokes equation in what is technically a non-turbulent regime, since they restricted their attention to large eddies of size $k < \Lambda \ll k_d$ [4]. There is, however, evidence that scaling results obtained for the infrared spectrum are valid for larger wavenumbers [5]. Furthermore, most later RG schemes are generalizations of the FNS theory. FNS postulated Gaussian white noise correlations of the form

$$\langle f_\alpha(\mathbf{k}, \omega) f_\beta(\mathbf{k}', \omega') \rangle = (2\pi)^{d+1} F_0 k^{-y} D_{\alpha\beta}(\mathbf{k}) \delta(\mathbf{k} + \mathbf{k}') \delta(\omega + \omega') \quad (3)$$

where $y = 0$ represents random stirring on a macroscopic scale. We now note two important differences between RG as applied to turbulence and RG as applied to critical phenomena. First, time-dependence can not be ignored in non-equilibrium situations, so the variable ω of Fourier transformed time is non-trivial. We will eventually have to rescale time as well as space in order to restore the coarse-grained Navier Stokes equation to its original form. Second, the field $\mathbf{u}(\mathbf{k}, \omega)$ is not a perturbation of a Gaussian distribution. Instead, the starting point for perturbation, hence for a Wilsonian treatment of RG, is the Navier Stokes equation with its non-linear term removed:

$$G_0^{-1}(k, \omega)u_\alpha(\mathbf{k}, \omega) = f_\alpha(\mathbf{k}, \omega) \quad (4)$$

where $G_0 = (i\omega + \nu_0 k^2)^{-1}$ is the unrenormalized Green

*Electronic address: benjamin@physics.harvard.edu

function.

We then divide Fourier space into low-frequencies $0 < k < \Lambda/b$ and high frequencies $\Lambda/b < k < \Lambda$. Equation (2) then divides into two parts:

$$\begin{aligned} G_0^{-1}(k, \omega) u_\alpha^<(\mathbf{k}, \omega) &= f_\alpha^<(\mathbf{k}, \omega) + \\ \lambda M_{\alpha\beta\gamma}^<(\mathbf{k}) \int_{q, \omega'}^\Lambda &\left[u_\beta^<(\mathbf{q}, \omega') u_\gamma^<(\mathbf{k} - \mathbf{q}, \omega - \omega') + \right. \\ 2u_\beta^<(\mathbf{q}, \omega') u_\gamma^>(\mathbf{k} - \mathbf{q}, \omega - \omega') &+ \\ \left. u_\beta^>(\mathbf{q}, \omega') u_\gamma^>(\mathbf{k} - \mathbf{q}, \omega - \omega') \right] &\quad (5) \end{aligned}$$

and the same expression with the substitution “<” \leftrightarrow “>” on every superscript. The above notation means that, for example, $\mathbf{u}(\mathbf{k}, \omega) = \mathbf{u}^>(\mathbf{k}, \omega)$ if $\Lambda/b < k < \Lambda$. To eliminate high frequencies, we obtain a perturbation series for $\mathbf{u}^>$ in terms of f and $\mathbf{u}^<$. Starting with the zeroth-order expression $\mathbf{u}_0^> = G_0 \mathbf{f}^>$, we repeatedly substitute the n th-order expression for \mathbf{u} into Equation (5) to obtain the next order. The last step of coarse-graining is to average over high frequencies, which is simplified by the Gaussian nature of the random force. FNS showed that this reproduces Equation (2) with a random force, in terms of $\mathbf{u}^<$, but with the addition of a triple product term:

$$\begin{aligned} 2\lambda^2 M_{\alpha\beta\gamma}^<(\mathbf{k}) \int d^3 q_1 d^3 q_2 M_{\beta\rho\sigma}^>(\mathbf{k} - \mathbf{q}_1) \times \\ G_0(\mathbf{k} - \mathbf{q}_1) u_\gamma^<(\mathbf{k} - \mathbf{q}_1) u_\rho^<(\mathbf{q}_2) u_\sigma^<(\mathbf{k} - \mathbf{q}_1 - \mathbf{q}_2) \end{aligned} \quad (6)$$

This term is negligible in the $k \rightarrow 0$ limit, and thus the FNS theory is at least a valid theory of the renormalized small k spectrum. There is some reason to believe that the coefficient of the triple nonlinear term may be neglected, in which case FNS would apply rigorously to the complete turbulent spectrum $k_i \ll k \ll k_d$, but this has only been shown to first order in perturbation [6]. It has been shown that some new terms generated at higher orders of λ are actually marginal operators [7]. Neglecting this term, the Navier-Stokes equation in terms of low-frequencies is recovered, in terms of coarse-grained parameters ν , λ , and F . Rescaling k and ω and renormalizing the velocity field restores the original form of the equation. The recursion relations show that for negative $\epsilon = 2 - d$, the only stable fixed point is $\lambda = 0$ and $\nu = \nu^* > 0$. Using the fact that velocity field correlations $u_\alpha(\mathbf{k}, \omega) u_\beta(\mathbf{k}', \omega')$ must be the same before and after RG, FNS derived a homogeneity relation that implied the K41 scaling law.

III. KOLMOGOROV’S CONSTANT

Yakhot and Orszag extended the FNS approach to extract the constant K_K of the K41 theory [5]. They assumed from the outset the correctness of the K41 scaling law, and thus made the choice of $y = d = 3$ in Equation (3) in order to reproduce the Kolmogorov energy cascade. In addition to presupposing the K41 theory, this method also relies on the postulate that a random force with the same statistical properties as the (deterministic)

long-wavelength modes of the Navier-Stokes equation will yield the correct energy spectrum. Yakhot and Orszag repeated the FNS analysis for $y =$ and obtained a non-zero fixed point for the nonlinear coupling parameter λ , and a wavenumber-dependent renormalized viscosity

$$\nu(k) = 0.4217 [2F_0 S_d (2\pi)^{-d}]^{1/3} k^{-4/3} \quad (7)$$

at the fixed point. They also calculated an energy spectrum

$$E(k) = 1.186 [2F_0 S_d (2\pi)^{-d}]^{2/3} k^{-5/3} \quad (8)$$

by replacing G_0 by the renormalized Green function $G(\mathbf{k}, \omega) = (i\omega + \nu^* k^2)$ in the lowest-order expression, Equation (4). Determination of K_K is then equivalent to relating F to observables. With a field-theoretical result due to Kraichnan [8] that for

$$\nu(k) = N\epsilon^{1/3} k^{-4/3} \text{ and } E(k) = K_K \epsilon^{2/3} k^{-5/3} \quad (9)$$

$N = 0.1904 K_K^2$, Equations (7) and (8) yield $K_K = 1.617$. Note that the constant F_0 is not arbitrary, since the above equations relate it to the dissipation rate ϵ .

The use of RG as above gives some welcome analytic results to a field scarce in them. Despite the agreement with experimental and numerical [2] values of K_K , the approaches we have so far discussed rely on several assumptions. In addition to the aforementioned fact that the discarded triple product term is not irrelevant in the RG sense, Yakhot and Orszag’s modification of the FNS theory uses an ϵ -expansion with $\epsilon = 4$ [9]. To address these issues, McComb and Watt renormalized the Navier-Stokes equation without invoking a random force [10]. Their idea was to treat as a statistical ensemble the set of all $\mathbf{u}^>$ that are consistent with a given state $\mathbf{u}^<$ according to the Navier-Stokes equation, *if there is some unobservable uncertainty in $\mathbf{u}^<$* . That is, high frequencies are only random in the sense of deterministic chaos. They then chose this ensemble to satisfy certain desirable properties under the operation of average over $\mathbf{u}^>$, the most important of which is

$$\langle u_{\alpha_1}^<(\mathbf{k}_1) \dots u_{\alpha_n}^<(\mathbf{k}_n) \rangle^> = u_{\alpha_1}^<(\mathbf{k}_1) \dots u_{\alpha_n}^<(\mathbf{k}_n) \quad (10)$$

The details of their derivation are long and too algebraically intense to present here. McComb and Watt had to implement their RG procedure numerically, with discrete removal of large-wavenumber shells rather than a differential RG flow of parameters. They followed analogous steps as in the papers previously mentioned to find a fixed point renormalized viscosity and nonlinear coupling, and came up with $K_K = 1.60 \pm 0.01$.

Park and Deem found another way to modify the FNS method so that it applies to the full range $k_i \ll k \ll k_d$ and to make its ϵ -expansion more rigorous [11]. They chose the random force to be proportional to the gradient of the velocity field:

$$f_\alpha(\mathbf{x}, t) = -F_{\alpha\beta\gamma}(\mathbf{x}) \partial_\beta u_\gamma(\mathbf{x}, t) \quad (11)$$

where

$$\langle F_{\alpha\beta\gamma}(\mathbf{k}) F_{\rho\sigma\tau}(\mathbf{k}') \rangle = (2\pi)^d \delta(\mathbf{k} + \mathbf{k}') k^{-y} D_{\alpha\beta\gamma}^{\rho\sigma\tau} \quad (12)$$

for

$$D_{\alpha\beta\gamma}^{\rho\sigma\tau} = D^{(1)}\delta_{\alpha\bar{\rho}}\delta_{\beta\bar{\sigma}}\delta_{\gamma\bar{\tau}} + D^{(2)}(\delta_{\alpha\beta}\delta_{\gamma\bar{\rho}}\delta_{\sigma\bar{\tau}} + \delta_{\alpha\gamma}\delta_{\beta\bar{\rho}}\delta_{\sigma\bar{\tau}}) \quad (13)$$

where we have introduced the notation that each term a symmetric sum over all permutations of barred variables. After decomposing the field into $\mathbf{u}^<$ and $\mathbf{u}^>$, Park and Deem carried out field-theoretic calculation of coarse-grained parameters to first-order in the coupling strength λ . They followed the standard procedure of rescaling space by $\mathbf{k}' = b\mathbf{k}$ and time by $t' = b^{-z}t$ (the symbol “z” is standard in the literature). Dimensional analysis requires the fields to renormalize as $\mathbf{u}'(\mathbf{k}', t') = b^{z-1-d}\mathbf{u}(\mathbf{k}, t)$. At this stage in the calculation, y and z are unknown, but they are fixed by two conditions. First, the K41 scaling behavior $E(k) \sim k^{-5/3}$ must hold. Second, the expected magnitude of $\partial_t \mathbf{u} = i\omega \mathbf{u}(\mathbf{k}, \omega)$ must be unaffected by RG. Note that rescaling of the time variable can be neglected in equilibrium situations because $\omega = 0$, so the second condition is satisfied trivially.

Park and Deem showed that in three dimensions, the RG flow cause $D^{(1)}$ and λ to vanish. That $\lambda^* = 0$ in their scheme is significant because λ , as we have seen, is the coefficient of the troublesome nonlinear term that was a marginal operator in other analyses. They found that $E(k) \sim k^{2z-3}$, which specifies $z = 2/3$ in order to match K41. This then specifies the force constant y . Using

Equation (9) and calculating $\nu(k)$ and $E(k)$ within their model, they obtained $K_K = 1.68$.

IV. A POSSIBLE MODEL

Most RG studies of turbulence share a common feature: they introduce a random force with an undetermined parameter, which is only set near the end of a calculation in order to make a derived energy spectrum match the K41 form. We consider here whether it is possible to adhere more transparently to the Kolmogorov spectrum from the outset of the calculation. We try to accomplish this by replacing the random force with random fluctuations in $\mathbf{u}^>$.

The idea is as follows: if we accept the K41 theory, we can model the energy transfer between wavenumbers as a cascade of constant rate local in k -space. We assume that a turbulent system is sufficiently chaotic such that all phase information is lost in interactions between different wavevectors. Since energy is proportional to $\langle \mathbf{u}^2 \rangle$, this amounts to the statement that for a fixed state $[\mathbf{u}^<]$, $[\mathbf{u}^>]^2$ is completely deterministic, whereas the direction of $\mathbf{u}^>$ is random.

V. CONCLUSION

-
- [1] Kolmogorov, A. *Dokl. Akad. Nauk. SSSR* **32**, 16 (1941)
 - [2] Verma, M. *arXiv nlin.CD*, 0510069v1 (2005)
 - [3] McComb, W. *The Physics of Fluid Turbulence* Oxford University Press (1990)
 - [4] Forster, D., Nelson, D. and Stephen, M. *Phys. Rev. A* **16**, 732 (1977)
 - [5] Yakhot, V. and Orszag, S. *App. Comp. Math.* **57**, 1722 (1986)
 - [6] DeDominicis, C. and Martin, P. *Phys. Rev. A* **19**, 419 (1979)
 - [7] Eyink, G. *Phys. Fluids* **6** 3063 (1994)
 - [8] Kraichnan, R. *J. Fluid Mech.* **47**, 525 (1971)
 - [9] Smith, L. and Woodruff, S. *Ann. Rev. Fluid Mech.* **30**, 275 (1998)
 - [10] McComb, W. and Watt, A. *Phy. Rev. A* **46**, 4797 (1992)
 - [11] Park, J. and Deem, M. *arXiv cond-mat.stat-mech*, 0202456v2 (2003)