RG treatment of an Ising chain with long range interactions

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An analysis of a one-dimensional Ising model with $1/r^\omega$ interactions is conducted using renormalization group techniques. An explicit example for a rescaling factor $b = 3$ is presented, including values for T_c and the critical exponent ν. Results for general b, in particular the limiting case of $b \to \infty$, are used to demonstrate the existence of phase transitions at finite temperature $T_c(\omega) \neq 0$ for $1 < \omega \leq 2$.

I. INTRODUCTION

The Ising model is one of the fundamental systems in statistical mechanics. The model is useful for multiple reasons; it's simplicity makes it an important pedagogical example and allows for efficient numerical simulation, yet it is also able to model various types of physical phenomena, such as ferromagnetism, lattice gases, and even neuron activity in the brain. The most familiar Ising models are those with ferromagnetic, nearest-neighbor interactions in one and two dimensions. Both cases can be solved exactly, using either renormalization group or transfer matrix methods (see, for example, [1]). The three-dimensional, nearest-neighbor Ising model remains unsolved, as do most one- and two-dimensional models with more complicated interactions.

An important category of Ising models are ferromagnetic models with long range interactions, by which we mean a d-dimensional lattice of N spin-1/2 Ising spins $\sigma_i = \pm 1$ subject to the Hamiltonian

$$\mathcal{H} = -J \sum_{(i,j)} \frac{1}{r_{ij}^\omega} \sigma_i \sigma_j. \tag{1}$$

Here ω and J are positive constants, with (i,j) denoting distinct pairs of lattice sites, and r_{ij} is the distance between sites i and j. We take r_{ij} to be measured in units of the lattice spacing a. The $\omega \to \infty$ limit yields the nearest-neighbor Ising model, while for $0 \leq \omega \leq d$, it can be shown that the system exhibits nonextensive behavior and accordingly the thermodynamic limit is not well-defined. We shall restrict to the case where $\omega > d$; nonextensive systems are discussed in the references [2] and [3].

In this letter, we consider the one-dimensional model of Eq. 1. In this case, we have $r_{ij} = |i - j|$. We study the system using a real-space renormalization group (RG) scheme using the Niemeijer-van Leeuwen (NvL) cumulant expansion (see [1]) to obtain the critical temperature T_c as well as the critical exponent ν governing the decay of the correlation length. The paper is outlined as follows;

first, we explicitly perform RG for a rescaling factor of $b = 3$ and obtain T_c and ν as functions of ω. We then generalize to arbitrary b, and in particular, we look at the limit $b \to \infty$ to obtain exact results.

II. NVL CUMULANT EXPANSION FOR $b = 3$

The partition function for the system in consideration is given by

$$Z = \sum_{\{\sigma_i\}=\pm 1} e^{-\beta\mathcal{H}[\{\sigma_i\}]} = \sum_{\{\sigma_i\}=\pm 1} \exp\left[K \sum_{(i,j)} \frac{\sigma_i \sigma_j}{r_{ij}^\omega}\right], \tag{2}$$

where we have introduced the dimensionless coupling $K = \beta J = J/k_B T$. To perform RG, we group the spins into cells α of size $b = 3$, as depicted in Fig. 1. The spins σ'_α are defined using the majority rule

$$\sigma'_\alpha = \text{sign}[\sigma_\alpha^1 + \sigma_\alpha^2 + \sigma_\alpha^3]. \tag{3}$$

The new spins $\{\sigma'_\alpha\}$ are well-defined and two-valued, since b is odd. We construct the renormalized interactions between the σ'_α by summing over the internal spins $\{\sigma_\alpha^i\}$, i.e.

$$e^{-\beta\mathcal{H}'[\{\sigma'_\alpha\}]} = \sum_{\{\sigma_\alpha^i\}=\pm 1} \delta_{\sigma_\alpha^i,\sigma'_\alpha} e^{-\beta\mathcal{H}[\{\sigma_\alpha^i\}]}, \tag{4}$$

where $\delta_{\sigma_\alpha^i,\sigma'_\alpha}$ is 1 if $\text{sign}[\sigma_\alpha^1 + \sigma_\alpha^2 + \sigma_\alpha^3] = \sigma'_\alpha$ and is -1 otherwise.

The basis for the NvL expansion is to separate the Hamiltonian into two parts $\mathcal{H} = \mathcal{H}_0 + \mathcal{V}$, where $\mathcal{H}_0$ contains only intracell interactions while $\mathcal{V}$ contains the intercell interactions. We then treat $\mathcal{V}$ perturbatively. We first define the expectation value

$$\langle \mathcal{O} \rangle_0 = \frac{1}{Z_0} \sum_{\{\sigma_\alpha^i\}=\pm 1} \delta_{\sigma_\alpha^i,\sigma'_\alpha} e^{-\beta\mathcal{H}_0[\{\sigma_\alpha^i\}]} \mathcal{O}, \tag{5}$$

where the unperturbed partition function Z_0 is defined as

$$Z_0[\{\sigma'_\alpha\}] = \sum_{\{\sigma_\alpha^i\}=\pm 1} \delta_{\sigma_\alpha^i,\sigma'_\alpha} e^{-\beta\mathcal{H}_0[\{\sigma_\alpha^i\}]}. \tag{6}$$
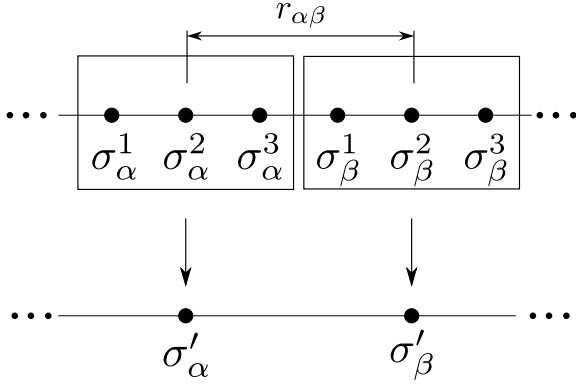
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FIG. 1: Grouping of spins into blocks of size $b = 3$.

We note that Z_0 depends on the configuration of the σ'_α. With these definitions, we can write Eq. 4 as

$$e^{-\beta\mathcal{H}'[\{\sigma'_\alpha\}]} = Z_0 \left[1 - \langle\beta\mathcal{V}\rangle_0 + \frac{1}{2}\langle(\beta\mathcal{V})^2\rangle_0 - \cdots \right]$$
$$= Z_0 \langle e^{-\beta\mathcal{V}}\rangle_0. \qquad (7)$$

From this we obtain the renormalized Hamiltonian:

$$-\beta\mathcal{H}'[\{\sigma'_\alpha\}] = -\ln Z_0[\{\sigma'_\alpha\}] + \ln\langle e^{-\beta\mathcal{V}}\rangle_0. \qquad (8)$$

For the Hamiltonian used in Eq. 2, $\mathcal{H}_0$ is given by

$$-\beta\mathcal{H}_0 = K\sum_\alpha \left[\sigma_\alpha^1\sigma_\alpha^2 + \sigma_\alpha^2\sigma_\alpha^3 + \frac{1}{2^\omega}\sigma_\alpha^1\sigma_\alpha^3 \right]. \qquad (9)$$

The corresponding partition function is then

$$Z_0[\{\sigma'_\alpha\}] = \prod_\alpha \left[e^{2K+K/2^\omega} + 2e^{-K/2\omega} + e^{-2K+K/2\omega} \right]$$
$$= \left[e^{2K+K/2^\omega} + 2e^{-K/2\omega} + e^{-2K+K/2\omega} \right]^{N/3}$$
$$= Z_{0,\alpha}^{N/3}, \qquad (10)$$

where we have defined the partition function for a particular block $Z_{0,\alpha}$ for simplicity. Note that Z_0 does not depend explicitly on σ'_α, and thus the $\ln Z_0$ in Eq. 8 is simply an additive constant. The first term in the cumulant expansion Eq. 8 is simply

$$\beta\langle\mathcal{V}\rangle_0 = K\sum_{(\alpha,\beta)} \sum_{\substack{i\in\alpha \\ j\in\beta}} \frac{1}{r_{ij}^\omega} \langle\sigma_\alpha^i\sigma_\beta^j\rangle_0, \qquad (11)$$

where the sum over (α,β) indicates summing over distinct pairs of blocks and i, j label the spins in blocks α, β respectively. Since the weight of the expectation values in Eq. 5 is a block-independent probability distribution, we have $\langle\sigma_\alpha^i\sigma_\beta^j\rangle_0 = \langle\sigma_\alpha^i\rangle_0\langle\sigma_\beta^j\rangle_0$. The averages $\langle\sigma_\alpha^i\rangle_0$ will clearly be independent of α, but will depend on i, due to

the $1/r^\omega$ interactions; a simple computation yields:

$$\langle\sigma_\alpha^1\rangle_0 = \langle\sigma_\alpha^3\rangle_0 = \frac{\sigma'_\alpha}{Z_{0,\alpha}} \left(e^{2K+K/2^\omega} + e^{-2K+K/2^\omega} \right)$$
$$\langle\sigma_\alpha^2\rangle_0 = \frac{\sigma'_\alpha}{Z_{0,\alpha}} \left(e^{2K+K/2\omega} + 2e^{-K/\omega} - e^{-2K+K/2^\omega} \right). \qquad (12)$$

We can simplify notation by writing $\langle\sigma_\alpha^i\rangle_0 = a_i(K)\sigma'_\alpha/Z_{0,\alpha}$, with $a_i(K)$ given by the coefficients above. We now let $r_{\alpha\beta}$ be the distance between the center sites, i.e. the distance between spins σ_α^2 and σ_β^2, in units of the rescaling length $b = 3$. The distances r_{ij}, between the spins σ_α^i and σ_β^j, then can take one of five values: $3r_{\alpha\beta}, 3r_{\alpha\beta} \pm 1$, or $3r_{\alpha\beta} \pm 2$. Thus, for $r_{\alpha\beta} \gg 1$, we can approximate

$$r_{ij} \approx 3r_{\alpha\beta}. \qquad (13)$$

This approximation has the effect of weakening the local interactions between blocks; for example, the worst case is when $r_{ij} = 1$, so according to Eq. 13, we are approximating $r_{ij}^\omega = 1$ by 3^ω, and therefore the nearest-neighbor interactions between the spins at the edges of the block are weighted by $1/3^\omega$ rather than 1. Since the approximation only differs drastically for the local interactions between blocks, there should not be a significant effect on the properties of the critical phenomena, which are excitations of long-wavelength modes of the system.

With this, we can now evaluate the sums in Eq. 11, obtaining

$$\beta\langle\mathcal{V}\rangle_0 = K_3' \sum_{(\alpha,\beta)} \frac{1}{r_{\alpha\beta}^\omega} \sigma'_\alpha\sigma'_\beta, \qquad (14)$$

where K_3' is defined as

$$K_3'(K) = \frac{K}{Z_{0,\alpha}^2 b^\omega} \left[\sum_{i=1}^3 a_i(K) \right]^2$$
$$= \frac{K}{Z_{0,\alpha}^2 b^\omega} \left[3e^{2K+K/2\omega} + 2e^{-K/2\omega} + e^{-2k+K/2\omega} \right]^2. \qquad (15)$$

Therefore, to first order in $\mathcal{V}$, the renormalized Hamiltonian is given by

$$-\beta\mathcal{H}' = -\frac{N}{3} \ln Z_{0,\alpha} + K' \sum_{(\alpha,\beta)} \frac{1}{r_{\alpha\beta}^\omega} \sigma'_\alpha\sigma'_\beta. \qquad (16)$$

As desired, $\beta\mathcal{H}'$ has the same form as $\beta\mathcal{H}$.

We now seek to determine the critical points of the system by analyzing the fixed points of the recurrence relation Eq. 15. A fixed point K^* has $K^* = K_3'(K^*)$. A plot of K_3' as a function of K is shown in Fig. 2, where the intersections of the curves with the straight line indicate fixed points. The values of K^* can be calculated numerically and are shown in Table I for a few select ω.
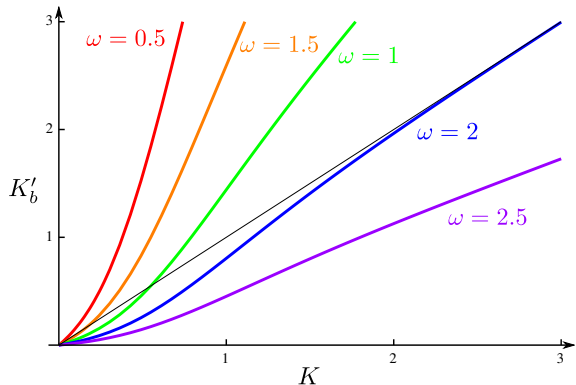
FIG. 2: Plot of recurrence relation Eq. 15 for the dimensionless coupling K. The black line corresponds to $K_b' = K$.

ω	K_c	K_c^{-1}	ν
1	0.166	6.02	4.26
1.25	0.337	2.97	2.58
1.5	0.549	1.82	2.17
1.75	0.873	1.15	2.42
1.99	2.20	0.455	18.6

TABLE I: Approximate values of K_c, K_c^{-1}, and ν for various ω, with a rescaling length of $b = 3$.

From Eq. 15, it is clear that for every ω there exist the two trivial fixed points at $K = 0$ ($T = \infty$) and $K = \infty$ ($T = 0$). As can be seen from Fig. 2, there exists some ω_c such that for $\omega < \omega_c$, $\mathrm{d}K_3'/\mathrm{d}K$ is greater than one at $K = 0$, implying that $K = 0$ is a repulsive fixed point, and thus $T_c = \infty$.

However, if $\omega > \omega_c$, then $\mathrm{d}K_3'/\mathrm{d}K$ is less than one at $K = 0$, implying that $K = 0$ is an attractive fixed point. Furthermore, since K_3' scales as $K_3' \sim 3^{2-\omega}K$ for $K \gg 1$, if $\omega_c < \omega < 2$, there exists another fixed point at finite $K = K_c(\omega) > 0$. Since the fixed point at $K = 0$ is attractive and the monotonicity of K_3' as a function of K requires that K_3' can cross K only once at finite K, $K_c(\omega)$ must be a repulsive fixed point, indicating a phase transition at a finite temperature. For $\omega > 2$, there is no finite K fixed point due to the asymptotic scaling of K_3', so $K = \infty$ must be a repulsive fixed point and therefore $T_c = 0$.

The exponent ν governing the divergence of the correlation length can be calculated from

$$\left.\frac{\mathrm{d}K_3'}{\mathrm{d}K}\right|_{K_c} = b^{1/\nu} \Rightarrow \nu(\omega) = \frac{\ln 3}{\ln(\mathrm{d}K_3'/\mathrm{d}K|_{K_c})}. \qquad (17)$$

The results from calculating $\nu(\omega)$ numerically are shown in Table I.

III. RESULTS FOR ARBITRARY b

We now consider performing the same RG transformation for $b = 2k + 1$, where k is an arbitrary positive integer. Since b is odd, the majority rule Eq. 3 can be generalized to $\sigma_\alpha' = \mathrm{sign}(\sum_{i=1}^b \sigma_\alpha^i)$. Generalizing to larger b is of interest because the nth cumulant of $\langle e^{-\beta\mathcal{V}}\rangle_0$ will be of order $b^{-n\omega}$; therefore, the cumulant expansion is becomes a series expansion in powers of $b^{-\omega}$, implying that $\mathcal{H}' = \langle\mathcal{V}\rangle_0$ is a better approximation for b large.

The general formalism of the previous section generalizes in the obvious way; to first order in the cumulant expansion we have the following generalization of Eq. 11

$$\mathcal{H}' = \langle\mathcal{V}\rangle_0 = \sum_{(\alpha,\beta)} \sum_{\substack{i\in\alpha \\ j\in\beta}} \frac{1}{r_{ij}^\omega} \langle\sigma_\alpha^i\rangle_0 \langle\sigma_\beta^j\rangle_0, \qquad (18)$$

except here i, j range from 1 to b, labeling the b spins in the block α, β respectively. Similarly, the averages in Eq. 18 have the form $\langle\sigma_\alpha^i\rangle_0 = a_i(K)\sigma_\alpha'$, where $a_i(K)$ does not depend on the block α. Following Eq. 13, we approximate $r_{ij} \approx br_{\alpha\beta}$. With this, the recursion relation Eq. 15 generalizes to

$$K_b'(K) = \frac{K}{Z_{0,\alpha}^2 b^\omega} \left[\sum_{i=1}^b a_i(K)\right]^2. \qquad (19)$$

The discussion of fixed points proceeds as before; it is easy to see that for $K \gg 1$, $a_i(K)/Z_{0,\alpha} \sim 1$ for all i (see appendix), and therefore Eq. 19 implies that $K_b'(K \gg 1) \sim b^{2-\omega}K$. There are still two trivial fixed points $K = 0$ and $K = \infty$, and there exists an $\omega_c(b)$ such that $K = 0$ is repulsive if $\omega < \omega_c(b)$ and attractive if $\omega > \omega_c(b)$. If $\omega_c(b) < \omega < 2$, then by monotonicity and the asymptotic scaling, there must exist a fixed point at finite K.

We now calculate $\omega_c(b)$. $\omega_c(b)$ is defined by the following condition

$$\left.\frac{\mathrm{d}K_b'}{\mathrm{d}K}\right|_{K=0} = Z_{0,\alpha}^{-2}(K=0)b^{-\omega_c} \left[\sum_{i=1}^b a_i(K=0)\right]^2 = 1. \qquad (20)$$

As shown in the appendix, $a_i(0)/Z_{0,\alpha}$ is given, for all i, by

$$\frac{a_i(K=0)}{Z_{0,\alpha}(K=0)} = \gamma(b) = \frac{(b-1)!}{2^{b-1}(\frac{b-1}{2}!)^2}. \qquad (21)$$

With this, Eq. 20 gives

$$1 = b^{2-\omega_c}\gamma^2(b) \Rightarrow \omega_c = 2\left[1 + \frac{\ln\gamma(b)}{\ln b}\right]. \qquad (22)$$

For $b = 3$, this gives $\omega_c(3) \approx 0.738$, consistent with the results in Table I. Using Stirling's formula for $n!$ as $n \to \infty$, we see that $\gamma(b) \propto b^{-1/2}$ for b large and thus $\omega_c \to 1$ in the limit $b \to \infty$. Therefore, we expect that the

exact solution to the model exhibits a phase transition at finite temperature for $1 < \omega < 2$, a result proven rigorously in [4].

As $\omega \to 2$ from below, we expect that $K_c \to \infty$. In this limit, expanding Eq. 19 yields (see appendix)

$$K'_b(K) \sim \frac{K}{b^\omega} \left[b - 4e^{-B(b)K} \right]^2, \tag{23}$$

where $B(b) = 2\sum_{n=1}^{b-1} 1/n^2$. As discussed in the appendix, $B(b)$ is the energy difference between the ground state and the first excited state of a single block of b spins. Rearranging this expression, we obtain

$$2 - \alpha \sim A(b)e^{-B(b)K_c}, \tag{24}$$

where $A(b) = 8/(b \ln b)$. As $b \to \infty$, we have $A \to 0$ and $B \to \pi^2/3$. Since $A(b)$ governs the region where Eq. 24 holds and $A \to 0$ as $b \to \infty$, Eq. 24 suggests that there is a phase transition at a finite temperature for $\omega = 2$, i.e. $T_c(\omega = 2) \neq 0$.

IV. CONCLUSION

In summary, we have seen that there exist three regimes for the one-dimensional Ising model with $1/r^\omega$ interactions:

(i) $\omega \leq 1$: The system is non-extensive and traditional statistical mechanics cannot be used,

(ii) $1 < \omega \leq 2$: The system exhibits a phase transition at finite $T_c(\omega) \neq 0$,

(iii) $\omega > 2$: The phase transition occurs at $T_c = 0$.

These results were obtained by studying the RG recurrence relations in the $b \to \infty$ limit, and reproduce the exact results, providing confidence in the methods used.

With more sophisticated analysis, it is possible to extract an estimate of $K_c^{-1}(\omega = 2) = \pi^2/12 \approx 0.823$; such a method is demonstrated in [5]. This approximation falls within the numerical estimate of $K_c^{-1} = 0.79 \pm 0.05$, given in [6]. The $\omega = 2$ case is particularly interesting as it can be mapped into the spin-1/2 Kondo problem, a model of electrical resistivity [6].

The methods used in this letter, modeled after those used in [5], could be generalized to higher dimensions or more complex interactions, such as the one-dimensional Potts model with long range interactions [7].

Appendix

In this section, we demonstrate how the values for $\gamma(b)$ and the asymptotic scaling form of $K'_b(K)$, Eq. 21 and Eq. 23 respectively, were obtained. To simplify some of the equations, we recall our definition of k, $b = 2k + 1$.

To obtain Eq. 21, we first find the form of $Z_{0,\alpha}$ at $K = 0$. For a given σ'_α, there are $k = (b-1)/2$ possible arrangements of the spins σ^i_α: one arrangement where $\sigma^i_\alpha = \sigma'_\alpha$ for all i in block α, $\binom{b}{1}$ arrangements with one spin opposite of σ'_α, $\binom{b}{2}$ arrangements with two spins opposite, etc. Since $K = 0$, each configuration gives the same weight $e^0 = 1$, so we have

$$Z_{0,\alpha}(K = 0) = \sum_{i=0}^{k} \binom{2k+1}{i} = 4^k = 2^{b-1}. \tag{25}$$

We now need the form of $a_i(K = 0)$, which is independent of i (since $K = 0$) and given by the difference between the number of configurations with $\sigma^i_\alpha = \sigma'_\alpha$ and the number of configurations with $\sigma^i_\alpha = -\sigma'_\alpha$. Consider the arrangement with i spins opposite of σ'_α. There are $\binom{b}{i}$ possible configurations of this form. $\binom{b-1}{i-1} = \binom{2k}{i-1}$ of these arrangements have σ^i_α opposite of $-\sigma'_\alpha$. Therefore, using some standard factorial identities (see, for example, [8]), we have

$$\frac{a_i(K = 0)}{Z_{0,\alpha}(K = 0)} = \sum_{i=0}^{k} \binom{2k+1}{i} - 2\sum_{i=0}^{k-1} \binom{2k}{i}$$

$$= -4^k + \binom{2k}{k}\left(1 + \frac{\sqrt{\pi}k!}{(k-1/2)!}\right)$$

$$= \frac{(2k)!}{(k!)^2} = \frac{(b-1)!}{(\frac{b-1}{2}!)^2}. \tag{26}$$

This combines with Eq. 25 to yield Eq. 21.

We now obtain the asymptotic expression Eq. 23 for $K'_b(K)$. We consider only the two lowest energy terms in $Z_{0,\alpha}$ and $a_i(K)$. The ground state is the unique configuration with all spins aligned with σ'_α, while the first excited state has two possible configurations: $\sigma^1_\alpha = -\sigma'_\alpha$ or $\sigma^b_\alpha = -\sigma'_\alpha$, while the rest of the spins are aligned with σ'_α. The difference in energy between these states is given by $\beta(E_0 - E_1) = -2K \sum_{i=1}^{b-1} 1/2^\omega$, where E_0 denotes the ground state energy and E_i denotes the energy of the ith excited state. Therefore, the first two terms of $Z_{0,\alpha}(K)$ are

$$Z_{0,\alpha}(K \gg 1) = e^{-\beta E_0}(1 + 2e^{-2K \sum 1/2^\omega} + \mathcal{O}(e^{\beta \Delta E_2})), \tag{27}$$

where $\Delta E_2 = E_0 - E_2$.

We look for a similar expansion of $a_i(K)$. If $i \neq 1, b$, then both of the first excited state configurations have $\sigma^i_\alpha = \sigma'_\alpha$, and therefore the coefficient of $e^{-\beta E_1}$ will have a factor of 2. However, if $i = 1, b$, then one configuration has $\sigma^i_\alpha = \sigma'_\alpha$ while the other has $\sigma^i_\alpha = -\sigma'_\alpha$; therefore the contributions from these two cancel, yielding

$$a_i(K) = \begin{cases} e^{-\beta E_0}(1 + 2e^{-2K \sum 1/2^\omega} + \mathcal{O}(e^{\beta \Delta E_2})) & i \neq 1, b \\ e^{-\beta E_0}(1 + \mathcal{O}(e^{\beta \Delta E_2})) & i = 1, b. \end{cases} \tag{28}$$

We note that for K large enough, $a_i(K)/Z_{0,\alpha} \sim 1$, as claimed in the previous section. Therefore, summing over the a_i and dividing by Eq. 27, we obtain

$$
\begin{aligned}
\frac{\sum a_i(K \gg 1)}{Z_{0,\alpha}(K \gg 1)} &= \frac{b + 2(b-2)e^{-2K\sum 1/2^\omega} + \mathcal{O}(e^{\beta\Delta E_2})}{1 + 2e^{-2K\sum 1/2^\omega} + \mathcal{O}(e^{\beta\Delta E_2})} \\
&= b - 4e^{-2K\sum 1/2^\omega} + \mathcal{O}(e^{\beta\Delta E_2}, e^{-4K}).
\end{aligned}
\tag{29}
$$

In the limit $\omega \to 2$, we can replace ω with two in the exponent, obtaining Eq. 23.

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