Fluctuations in Superconductors using Landau-Ginzburg Theory

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May 10, 2010

Abstract

In this paper, we study the fluctuations in superconductors in the framework of Landau-Ginzurg theory. First, we review the formulation of LG theory and derive the LG equations. The we calculate the fluctuation contribution to the saddle point heat capacity, both inside and outside the immediate vicinity of the transition point. Outside the immediate vicinity, we follow the Gaussian integral calculation done in class, and obtained the correction to the heat capacity. In the immediate vicinity of the transition point, instead of using the perturbative renormalization group method introduced in class, we use the multiplicative RG method introduced by Larkin. Through calculating the critical exponents, we prove that the two methods agree with each other to leading order in t. We conclude the paper by a calculation of the fluctuations of the magnetic field, and find that the fluctuations can change the order of the phase transition.

1 Formulation and LG Equations

The Landau-Ginzburg formalism is quite general. In class, we discussed the formalism in the context of a magnetic system. It is equally eligible to apply the formalism to a superconducting system, with the following Landau-Ginzburg functional free energy:

$$\mathcal{F}[\Psi(\mathbf{r})] = F_n + \int d^3\mathbf{r} \left(a\left|\Psi\right|^2 + \frac{b}{2}\left|\Psi\right|^4 + \frac{1}{4m}\left|\nabla\Psi\right|^2 + \dots \right) \tag{1}$$

where F_n is the free energy in the normal phase [6]. The above form is for the case when the external magnetic field is zero. For superconducting states inside an external field, we have:

$$\mathcal{F}[\Psi(\mathbf{r})] = F_n + \int d^3\mathbf{r} \left(a\left|\Psi\right|^2 + \frac{b}{2}\left|\Psi\right|^4 + \frac{1}{4m}\left|(-i\nabla - 2e\boldsymbol{A})\Psi\right|^2 + \frac{1}{8\pi}\left|\nabla\times\boldsymbol{A}\right|^2 + \dots \right) \tag{2}$$

where we have changed $\nabla \to \nabla - 2ie\boldsymbol{A}$ to make the derivative term gauge invariant. Also, the last term represents the magnetic field energy [2]. From the implicit coefficient $2e$ rather than e and mass term $4m$ rather than $2m$, we have already put in the idea of the "Cooper pair". More explicitly: from the microscopic BCS theory, we know that the order parameter Ψ has the form $\Psi = \sum_k \langle c^\dagger_{-k\downarrow} c_{k\uparrow}\rangle$. Under gauge transformation, we have $c(\mathbf{r}) \to e^{i\theta(\mathbf{r})}c(\mathbf{r})$ and $\Psi(\mathbf{r}) \to e^{2i\theta(\mathbf{r})}\Psi(\mathbf{r})$. Hence there is a factor of 2 [9].

This discussion also shows that our choice of order parameter Ψ here is nothing but the wavefunction of the Cooper pair. However, we will avoid discussion of the detailed BCS theory here. Now as a direct result of the Landau-Ginzburg functional equation 2, we derive the Landau-Ginzburg equations:

$$\begin{aligned}
\delta\mathcal{F} &= \int d^3\mathbf{r} \left(a\Psi\delta\Psi^* + b\left|\Psi\right|^2\Psi\delta\Psi^* + \frac{1}{4m}\left[(i\nabla - 2e\boldsymbol{A})\,\delta\Psi^* \right] \right) \\
&= \int d^3\mathbf{r} \left(a\Psi\delta\Psi^* + b\left|\Psi\right|^2\Psi\delta\Psi^* + \frac{1}{4m}\left[-2e\boldsymbol{A}\cdot\delta\Psi^*\mathbf{D}\Psi - i\nabla\cdot(\delta\Psi^*\mathbf{D}\Psi) \right] \right) \\
&= \int d^3\mathbf{r} \left(a\Psi + b\left|\Psi\right|^2\Psi + \frac{1}{4m}\big(\mathbf{D}\cdot\mathbf{D}\Psi\big) \right)\delta\Psi^* = 0 \\
&\Rightarrow \; a\Psi + b\left|\Psi\right|^2\Psi + \frac{1}{4m}\big(\mathbf{D}^2\Psi\big) = 0
\end{aligned} \tag{3}$$

Similarly,

$$\begin{aligned}
\delta\mathcal{F} &= \int d^3\mathbf{r} \left(\frac{-2ie}{4m}\big(\Psi\mathbf{D}^*\Psi^* - \Psi^*\mathbf{D}\Psi\big) - \frac{1}{4\pi}\partial_b\epsilon_{abc}\epsilon_{ade}\partial_d A_e \right) \\
&= \int d^3\mathbf{r} \left(-\frac{ie}{2m}\big(\Psi\nabla\Psi^* - \Psi^*\nabla\Psi\big) - \frac{2e^2}{m}\left|\Psi\right|^2\boldsymbol{A} - \frac{1}{4\pi}\nabla\times\nabla\times\boldsymbol{A} \right) = 0 \\
&\Rightarrow \nabla\times\nabla\times\boldsymbol{A} = 4\pi\left(-\frac{ie}{2m}\big(\Psi\nabla\Psi^* - \Psi * \nabla\Psi\big) - \frac{2e^2}{m}\left|\Psi\right|^2\boldsymbol{A} \right) = 4\pi\boldsymbol{j}_s
\end{aligned} \tag{4}$$

Equations 3 and 4 are the famous Landau-Ginzburg equations. As a direct application of the LG equations, we can derive the penetration depth for a superconductor in a magnetic field as follows [2]. Suppose the order parameter is homogeneous. We then have $\boldsymbol{j}_s = -\frac{1}{4\pi}\lambda^{-2}\boldsymbol{A}$, where $\lambda^{-2} \equiv \frac{8\pi e^2}{m}\left|\Psi\right|^2$.

Thus 4 is reduced to $\nabla\times\nabla\times\boldsymbol{A} = \lambda^{-2}\boldsymbol{A}$. Applying curl to both sides, we have:

$$\nabla^2\boldsymbol{B} = -\lambda^{-2}\boldsymbol{B} \tag{5}$$

which is the famous “London Equation”, with λ the penetration depth.

2 Fluctuations in heat capacity

2.1 Saddle point result

Assume the order parameter is homogeneous and perform a saddle point approximation as we did in class for magnetization:

$$|\bar{\Psi}| = \left\{ \begin{array}{cc} \sqrt{\frac{-a}{b}} & \text{for } a < 0 \\ 0 & \text{for } a > 0 \end{array} \right\} \tag{6}$$

The free energy is thus given by:

$$F = \mathcal{F}_{min} = \left\{ \begin{array}{cc} F_n - \frac{a^2}{2b} V & \text{for } a < 0 \\ F_n & \text{for } a > 0 \end{array} \right\} \tag{7}$$

$$C \equiv -T \frac{\partial^2 f}{\partial T^2} \propto -T_c \frac{\partial^2}{\partial a^2} \left(\frac{F}{V} \right) = \left\{ \begin{array}{cc} \frac{T_c}{b} & \text{for } a < 0 \\ 0 & \text{for } a > 0 \end{array} \right\} \tag{8}$$

As in class, we find a jump in the heat capacity.

2.2 Fluctuation contribution to heat capacity (outside immediate vicinity [4])

As done in class, we now calculate the fluctuation correction to our saddle point result. Note that the fluctuation contribution is meaningful as long as we do the calculation in the region $|t| \geq t_G \approx \left(\xi_0^d \Delta C_{sp} \right)^{\frac{2}{d-4}}$ as shown in class.

Before we talk about the general d dimensional case, let us first discuss a special case $d = 0$, where the exact solution is available.

2.2.1 Zero dimensional calculation [8] [5]

$$
\begin{aligned}
Z_0 &= \int d^2\Psi_0 \exp\left(-\frac{\mathcal{F}\left[\Psi_0\right]}{T}\right) = \pi \int d\left|\Psi_0\right|^2 \exp\left(-\frac{a\left|\Psi_0\right|^2 + \frac{b}{2V}\left|\Psi_0\right|^4}{T}\right) \\
&= \sqrt{\frac{\pi^2 VT}{2b}} \exp\left(x^2\right)\left(1 - \text{erf}\left(x\right)\right)\Big|_{x=a\sqrt{\frac{V}{2bT}}}
\end{aligned} \tag{9}
$$

Since this is an analytic function of the reduced temperature a, we see that the fluctuation has removed the phase transition in 0 dimensions.

2.2.2 General dimensional calculation

Following the fluctuation calculation doen in class, we now write the order parameter in the form $\Psi(\mathbf{r}) = \bar{\Psi} + \Phi(\mathbf{r})$, where $\bar{\Psi}$ is the equilibrium order parameter and $\Phi(\mathbf{r})$ is the fluctuation. Since $\bar{\Psi}$ is real, we can rewrite the fluctuation as:

$$
\Psi(\mathbf{r}) = \bar{\Psi} + \text{Re}\left[\Phi(\mathbf{r})\right] + \text{Im}\left[\Phi(\mathbf{r})\right] \tag{10}
$$

Thus, through expanding the terms in the Landau-Ginzburg functional, we have:

$$
\begin{aligned}
Z\left[\Psi\right] = \exp\left(-\frac{a\bar{\Psi}^2 + \frac{b}{2}\bar{\Psi}^4}{T}\right) \prod_k \int d\text{Re}\left[\phi_k\right] d\text{Im}\left[\phi_k\right] \\
\exp\left(-\frac{1}{T}\left[\left(3b\bar{\Psi}^2 + a + \frac{k^2}{4m}\right)\text{Re}\left[\phi_k\right]^2 + \left(b\bar{\Psi}^2 + a + \frac{k^2}{4m}\right)\text{Im}\left[\phi_k\right]^2\right]\right)
\end{aligned} \tag{11}
$$

Carrying out the Gaussian integrals, we get:

$$
F = -\frac{T}{2}\sum_k \left[\ln\left(\frac{\pi T_c}{3b\bar{\Psi}^2 + a + \frac{k^2}{4m}}\right) + \ln\left(\frac{\pi T_c}{b\bar{\Psi}^2 + a + \frac{k^2}{4m}}\right)\right] \tag{12}
$$

Defining the longitudinal and transverse correlation lengths

$$
\begin{aligned}
\frac{1}{4m\xi_l^2} &= a + 3b\bar{\Psi}^2 = \frac{1}{2}\frac{\partial^2 f\left[\Psi\right]}{\partial \mathrm{Re}\left[\phi\right]^2} = \left\{ \begin{array}{cc} -2a & \text{for } a<0 \\ a & \text{for } a>0 \end{array} \right\} \\
\frac{1}{4m\xi_t^2} &= a + b\bar{\Psi}^2 = \frac{1}{2}\frac{\partial^2 f\left[\Psi\right]}{\partial \mathrm{Im}\left[\phi\right]^2} = \left\{ \begin{array}{cc} 0 & \text{for } a<0 \\ a & \text{for } a>0 \end{array} \right\}
\end{aligned} \tag{13}
$$

We then have:

$$
\begin{aligned}
F &= \frac{T}{2}\sum_k \left[\ln\left(\frac{k^2+\xi_l^2}{4m\pi T_c}\right) + \ln\left(\frac{k^2+\xi_t^2}{4m\pi T_c}\right)\right] \\
C_{\text{singular}} &= -\frac{1}{VT_c}\left(\frac{\partial^2 F}{\partial a^2}\right) \propto \left\{ \begin{array}{cc} \frac{1}{V}\sum_k \frac{1}{\left(a+\frac{k^2}{4m}\right)^2} & \text{for } a>0 \\ \frac{4}{V}\sum_k \frac{1}{\left(2a-\frac{k^2}{4m}\right)^2} & \text{for } a<0 \end{array} \right\}
\end{aligned} \tag{14}
$$

Thus we can see that the tendency of C being divergent when approaching T_c. Notice that the discussion so far is only valid outside the immediate vicinity of T_c, where the fluctuation is small; to seriously discuss the divergence, we need to go to the next section.

2.3 Fluctuations in the immediate vicinity; Multiplicative RG [1] [5]

In the last section, we discussed the fluctuation outside the immediate vicinity of the critical temperature. In that region, the fluctuation is small. Thus, the starting point of the saddle point approximation is justified. Now, we proceed to calculate the fluctuation in the immediate vicinity, i.e. $\left|t\right| \leq t_G = \left(\xi_0^d \Delta C_{sp}\right)^{\frac{2}{d-4}}$. This can be achieved using the renormalization group (RG) method [4].

In class, we did the calculation using perturbative RG. The basic idea is to combine the perturbative series expansion method with the renormalization group approach. Here, we present a somewhat different method using only the renormalization group: the "multiplicative RG". This method was first known in quantum field theory in 1954 [1]. The two methods, as we will see by the end of the calcuation, are completely equivalent.

As we did in class, first we subdivide the fluctuations into two parts: $\Psi_{k>\Lambda}$ (fast) and $\Psi_{k<\Lambda}$ (slow). If Λ is large enough, we can treat the fast modes as Gaussian and integrate over them. Then, the functional only depends on the slow modes. Now we can do the whole process again.. Thus, through repeating the process, we can obtain the complete partition function.

The above process can be better understood through the calculation we did in section 2.2.2. Back then, we treated the saddle point value $\bar{\Psi}$ as the "slow mode" and the fluctuation part $\phi(\mathbf{r})$ as the "fast mode". The "fast mode" was then treated as Guassian, which we integrated over.

The central assumption we made here is that the free energy is of the same form at any subsequent step. This has been properly justified in class [4]. We will proceed to do the calculation using mathematical induction. After the $n-1^{\text{th}}$ step:

$$\mathcal{F}\left[\bar{\Psi}_{\Lambda_{n-1}}\right] = F_{N,\Lambda_{n-1}} + \int d^3\mathbf{r}\left(a_{\Lambda_{n-1}}\left|\bar{\Psi}_{\Lambda_{n-1}}\right|^2 + \frac{b_{\Lambda_{n-1}}}{2}\left|\bar{\Psi}_{\Lambda_{n-1}}\right|^4 + \frac{1}{4m}\left|\nabla\bar{\Psi}_{\Lambda_{n-1}}\right|^2\right) \tag{15}$$

Now write $\bar{\Psi}_{\Lambda_{n-1}} = \bar{\Psi}_{\Lambda_n} + \phi_{\Lambda_n}$. As long as we make the step small enough, Λ_{n-1} is close enough to Λ_n. Thus ϕ would be small, and using only a gaussian form would be justified. Using the property that for dimension close to 4, it is still possible to choose $\phi_{\Lambda_n} \ll \bar{\Psi}_{\Lambda_n}$ even when $\Lambda_n \gg \Lambda_{n-1}$. We then have from equations 11 and 12:

$$\begin{aligned}\mathcal{F}\left[\bar{\Psi}_{\Lambda_n}\right] =& F_{N,\Lambda_{n-1}} + \int d^3\mathbf{r}\left(a_{\Lambda_n}\left|\bar{\Psi}_{\Lambda_n}\right|^2 + \frac{b_{\Lambda_n}}{2}\left|\bar{\Psi}_{\Lambda_n}\right|^4 + \frac{1}{4m}\left|\nabla\bar{\Psi}_{\Lambda_n}\right|^2\right)\\ &- \frac{T}{2}\sum_{\Lambda_n<\left|\boldsymbol{k}\right|<\Lambda_{n-1}}\left[\ln\left(\frac{\pi T_c}{3b_{\Lambda_n}\left|\bar{\Psi}_{\Lambda_n}\right|^2 + a_{\Lambda_n} + \frac{k^2}{4m}}\right) + \ln\left(\frac{\pi T_c}{b_{\Lambda_n}\left|\bar{\Psi}_{\Lambda_n}\right|^2 + a_{\Lambda_n} + \frac{k^2}{4m}}\right)\right]\end{aligned} \tag{16}$$

Since we assumed the "fast modes" to be small, we can taylor expand the additional terms in equation 16:

$$
\begin{aligned}
\ln\left(\frac{\pi T_c}{3b\left|\bar{\Psi}\right|^2 + a + \frac{k^2}{4m}}\right) &= \ln\left(\frac{\pi T_c}{\left(a + \frac{k^2}{4m}\right)\left(1 + \frac{3b}{a+\frac{k^2}{4m}}\left|\bar{\Psi}\right|^2\right)}\right) \\
&\approx \ln\left(\frac{\pi T_c}{a + \frac{k^2}{4m}}\right) + \ln\left(1 - \frac{3b}{a + \frac{k^2}{4m}}\left|\bar{\Psi}\right|^2\right) \\
&\approx \ln\left(\frac{\pi T_c}{a + \frac{k^2}{4m}}\right) - \frac{3b}{a + \frac{k^2}{4m}}\left|\bar{\Psi}\right|^2 - \frac{9b^2}{\left(a + \frac{k^2}{4m}\right)^2}\left|\bar{\Psi}\right|^4
\end{aligned}
\tag{17}
$$

Similarly,

$$
\ln\left(\frac{\pi T_c}{b\left|\bar{\Psi}\right|^2 + a + \frac{k^2}{4m}}\right) \approx \ln\left(\frac{\pi T_c}{a + \frac{k^2}{4m}}\right) - \frac{b}{a + \frac{k^2}{4m}}\left|\bar{\Psi}\right|^2 - \frac{b^2}{\left(a + \frac{k^2}{4m}\right)^2}\left|\bar{\Psi}\right|^4
\tag{18}
$$

Combining equations 17 and 18 and plugging into equation 16, we have:

$$
\begin{aligned}
F_{n,\Lambda_n} &= F_{n,\Lambda_{n-1}} - T\sum_{\Lambda_n<\left|\boldsymbol{k}\right|<\Lambda_{n-1}} \ln\left(\frac{\pi T_c}{a_{\Lambda_n} + \frac{k^2}{4m}}\right) \\
a_{\Lambda_n} &= a_{\Lambda_{n-1}} + 2T\sum_{\Lambda_n<\left|\boldsymbol{k}\right|<\Lambda_{n-1}} \frac{b_{\Lambda_n}}{a_{\Lambda_n} + \frac{k^2}{4m}} \\
b_{\Lambda_n} &= b_{\Lambda_{n-1}} - 5T\sum_{\Lambda_n<\left|\boldsymbol{k}\right|<\Lambda_{n-1}} \frac{b_{\Lambda_n}^2}{\left(a_{\Lambda_n} + \frac{k^2}{4m}\right)^2}
\end{aligned}
\tag{19}
$$

From the above, we can get the recursion relations:

$$
\begin{aligned}
\frac{\partial F\left(\Lambda\right)}{\partial\Lambda} &= -T\mu_D\Lambda^{D-1}\ln\left(\frac{\pi T_c}{a\left(\Lambda\right) + \frac{\Lambda^2}{4m}}\right) \\
\frac{\partial a\left(\Lambda\right)}{\partial\Lambda} &= -2T\mu_D\frac{b\left(\Lambda\right)\Lambda^{D-1}}{a\left(\Lambda\right) + \frac{\Lambda^2}{4m}} \\
\frac{\partial b\left(\Lambda\right)}{\partial\Lambda} &= 5T\mu_D\frac{b\left(\Lambda\right)^2\Lambda^{D-1}}{\left(a\left(\Lambda\right) + \frac{\Lambda^2}{4m}\right)^2}
\end{aligned}
\tag{20}
$$

In the above equations, $\mu_D k^{D-1} dk = d^D k/(2\pi)^D$ is the reduced solid angle. From the above equations, we could also see that Λ has to be small so that we can change the sum to an integral.

The first consequence of equation 20 is the shift in critical temperature. Let us call the mean field transition temperature T_{c0}. This temperature corresponds to $a\left(T_{c0}, \Lambda \sim \xi^{-1}\right) = 0$, with $\Lambda \sim \xi^{-1}$.

After completing the renormalization procedure, we have: $a\left(T_c, \Lambda = 0\right) = 0$, where T_c is the true transition temperature. This can be calculated by integrating the second equation of 20:

$$a(\xi^{-1}) = a\delta T_c = \int_{a(0)}^{a(\xi^{-1})} da = -8mTb\mu_D \int_0^{\frac{1}{\xi}} d\Lambda \Lambda^{D-3} \tag{21}$$

This can be evaluated in 2D and 3D cases to obtain the shift in critical temperature:

$$\begin{aligned} \frac{\delta T_c^{3D}}{T_c} &\sim \frac{2mb}{\pi\xi} = -\frac{7\zeta(3)}{16\pi^3 \nu T_c \xi^3} \\ \frac{\delta T_c^{2D}}{T_c} &\sim 2\left[\frac{7\zeta(3)}{32\pi^3}\frac{1}{\nu_2 T_c \xi^2}\right] \ln\left(\frac{7\zeta(3)}{8\pi^3}\frac{1}{\nu_2 T_c \xi^2}\right) \end{aligned} \tag{22}$$

Details can be found in [5].

Now, proceed to solve equations 20 at T_c (the partial fixed point):

$$\begin{aligned} a(T_c, \Lambda) &= \frac{D-4}{4m[5+(4-D)]}\Lambda^2 \\ b(T_c, \Lambda) &= \left(\frac{5}{16m^2 T\mu_D}\right)\frac{4-D}{[5+(4-D)]^2}\Lambda^{4-D} \end{aligned} \tag{23}$$

The above solutions are obtained using power series. Thus it is a valid solution only for small enough Λ. It is argued in [5] that for D close to 4, it is possible to extend the validity of the above equations to the Landau-Ginzburg region; and through matching the results on the crossover, we can solve the above equations even further. This can be achieved through the following.

Let $\epsilon = 4 - D$ Since $a(\Lambda) \to 0$, we can omit the denominator in equation 20 and write:

$$\frac{\partial b}{(\ln\Lambda)^{-1}} = 5(4m)^2 T\mu_D \frac{b^2}{\Lambda^{4-D}} \tag{24}$$

which is the famous equation first obtained by Gell-Mann and Low in quantum field theory [1].

The solution of equation 24 is of the form:

$$b^{-1}(T_c, \Lambda) = b_0^{-1} + \frac{80m^2}{4-D} T\mu_D \left(\Lambda^{D-4} - \xi^{4-D}\right) \tag{25}$$

where we have used the matching condition at $\Lambda \sim \xi^{-1}$. Similarly, for $a(T, \Lambda)$, we could perform calculations as follows. Let

$$a(T, \Lambda) = a(T_c, \Lambda) + \alpha(T_c, \Lambda)t \tag{26}$$

where $a(T_c, \Lambda)$ is the one in equation 23. Now expand the second equation in 20 using 26 in powers of t:

$$\frac{\partial \alpha(T_c, \Lambda)}{\partial \Lambda} = 2T\mu_D \frac{b(T_c, \Lambda)\Lambda^{D-1}}{\left(a(T_c, \Lambda) + \frac{\Lambda^2}{4m}\right)^2} \alpha(T_c, \Lambda) \tag{27}$$

Same as before, omit $a(T_c, \Lambda)$ in the denominator of 27, we thus have:

$$\alpha(T_c, \Lambda) = \left[1 + \frac{80m^2 b_0}{4-D} T\mu_D(\Lambda^{D-4} - \xi^{4-D})\right]^{-\frac{2}{5}} \tag{28}$$

Now from the microscopic theory, define a special "coherence length"

$$\xi(T) = \frac{\xi}{\sqrt{tT_c}} \tag{29}$$

This "generalized" coherence length is determined by

$$\xi^{-2}(T) = 4m\alpha(T_c, \xi^{-1}(T))t \tag{30}$$

Setting $\Lambda = \xi^{-1}(T)$ and substituting 29 and 30 into 28, we eventually arrive at the self-consistent equation for $\xi(T)$:

$$\xi(T) = (4m)^{\frac{1-D}{2}} \frac{4-D}{20 b_0 T \mu_D \sqrt{T_c}} \left(\frac{t}{T_c}\right)^{-\nu} \tag{31}$$

and

$$\nu = \frac{1}{2\left(1 - \frac{4-D}{5}\right)} \approx \frac{1}{2} + \frac{4-D}{10} = \frac{1}{2} + \frac{\epsilon}{10} \tag{32}$$

This result can be compared with the one we calculated in class using perturbative RG [4] (c.f. equation IV.60).

$$\nu = \frac{1}{y_t} = \frac{1}{2} + \frac{1}{4}\left(\frac{n+2}{n+8}\right)\epsilon + \mathcal{O}(\epsilon^2) \tag{33}$$

Setting $n = 2$ (since the order parameter Ψ is complex), we have:

$$\nu = \frac{1}{2} + \frac{1}{10}\epsilon + \mathcal{O}(\epsilon^2) \tag{34}$$

which totally agrees with our result in 32, as expected.

We conclude this section by a calculation on the heat capacity. Taking the second derivative of the first equation in 20, we have:

$$\frac{\partial C(\Lambda)}{\Lambda} = T^2 \mu_D \frac{\Lambda^{D-1}\alpha^2(\Lambda)}{\left(\alpha(\Lambda)t + \frac{\Lambda^2}{4m}\right)^2} \tag{35}$$

Integrating the above equation over $\Lambda \leq \xi^{-1}$:

$$\begin{aligned} C &= \int_0^{\xi^{-1}(T)} d\Lambda T^2 \mu_D \frac{\Lambda^{D-1}\alpha^2(\Lambda)}{\left(\alpha(\Lambda)t + \frac{\Lambda^2}{4m}\right)^2} + \int_{\xi^{-1}(T)}^{\xi^{-1}} \frac{\Lambda^{D-1}\alpha^2(\Lambda)}{\left(\alpha(\Lambda)t + \frac{\Lambda^2}{4m}\right)^2} \\ &\approx \int_0^{\xi^{-1}(T)} d\Lambda T^2 \mu_D \frac{\Lambda^{D-1}\alpha^2(\Lambda)}{\left(\alpha(\Lambda)t + \frac{\Lambda^2}{4m}\right)^2} + \int_{\xi^{-1}(T)}^{\xi^{-1}} \frac{\Lambda^{D-1}\alpha(\Lambda)}{\left(\frac{\Lambda^2}{4m}\right)^2} \end{aligned} \tag{36}$$

Note that in the first integral, we need to use equation 23 for the value of $\alpha(\Lambda)$, while in the second integral, we can use 26 for the value of $\alpha(T_c, \Lambda)$. After the integration, the first term $\sim \epsilon^2$ and can thus be ignored. Thus to leading order:

$$C(\Lambda = 0) = \left[(4mT)^2 \mu_D\right]^{\frac{1}{5}} \left[\frac{4(4-D)}{5b_0}\right]^{4/5} \frac{5}{4-D} \xi^{\frac{4-D}{5}}(T) \tag{37}$$

Substituting in equation 31, we have:

$$\begin{aligned} C &= 2^{\frac{12}{5}} \left[5\mu_D \frac{m^2T^2}{b_0^4(4-D)}\right]^{\frac{1}{5}} \left(\frac{t}{T_c}\right)^{-\alpha} \\ \alpha &= \frac{4-D}{10\left[1 - \frac{4-D}{5}\right]} \approx \frac{\epsilon}{10} \end{aligned} \tag{38}$$

This result can also be compared with the one calculated in class (c.f. IV.61): $\alpha = \dfrac{4-n}{2(n+8)}\epsilon + \mathcal{O}(\epsilon^2)$. Setting $n = 2$, we have $\alpha = \dfrac{\epsilon}{10} + \mathcal{O}(\epsilon^2)$, which agrees with 37. Thus, the critical exponents are obtained using multiplicative RG. Of course, higher order calculations can be done [7]; but the leading order calculation is already a very good approximation, as seen in class.

3 Fluctuations of the magnetic field [5] [3]

All the discussions we had so far are under the condition that the magnetic field is a constant. Here we discuss the effect of fluctuations of a magnetic field. Assume now that the order parameter is homogenous. We then have from equation 2 (setting $\nabla \cdot \boldsymbol{A} = 0$):

$$\mathcal{F}[\boldsymbol{A}(\mathbf{r})] = F_n + \int d^3\mathbf{r} \left(a\left|\bar{\Psi}\right|^2 + \frac{b}{2}\left|\bar{\Psi}\right|^4 + \frac{e^2}{m}\left|\bar{\Psi}\right|^2 \boldsymbol{A}^2 + \frac{1}{8\pi}\left|\nabla \times \boldsymbol{A}\right|^2\right) \tag{39}$$

writing in Fourier space:

$$\begin{aligned} \mathcal{F}[\boldsymbol{A}(\mathbf{r})] &= F_n + V\left(a\left|\bar{\Psi}\right|^2 + \frac{b}{2}\left|\bar{\Psi}\right|^4\right) + \int d^3\mathbf{k} \left(+\frac{e^2}{m}\left|\bar{\Psi}\right|^2 \boldsymbol{A}^2 + \frac{1}{8\pi}k^2\boldsymbol{A}^2\right) \\ &= F_s\left[\bar{\Psi}, 0\right] + \sum_k \left(\frac{e^2}{m}\left|\bar{\Psi}\right|^2 + \frac{k^2}{8\pi}\right)\boldsymbol{A}_k^2 \end{aligned} \tag{40}$$

Thus the free energy is now:

$$
\begin{aligned}
F &= F_s\left[\bar{\Psi}, 0\right] + \ln \int \mathcal{D}\boldsymbol{A}_x \mathcal{D}\boldsymbol{A}_y \exp\left(-\frac{1}{T}\sum_k \left[\left(\frac{e^2\left|\bar{\Psi}\right|^2}{m} + \frac{k^2}{8\pi}\right)\boldsymbol{A}_k^2\right]\right) \\
&= F_s\left[\bar{\Psi}, 0\right] + T\sum_k \ln\left(\frac{\pi T}{\frac{e^2\left|\bar{\Psi}\right|^2}{m} + \frac{k^2}{8\pi}}\right)
\end{aligned}
\tag{41}
$$

Note that this calculation is completely analogous to the one we did for fluctuations in Ψ (see section 2.2). Now, expand the second term in equation 41:

$$
\begin{aligned}
\sum_k \ln\left(\frac{\pi T}{\frac{e^2\left|\bar{\Psi}\right|^2}{m} + \frac{k^2}{8\pi}}\right) &= \sum_k \ln\left(\frac{\pi T}{\frac{k^2}{8\pi}\left(1 + \frac{8\pi e^2\left|\bar{\Psi}\right|^2}{mk^2}\right)}\right) \\
&\approx \sum_k \left[\ln\left(\frac{8\pi^2 T}{k^2}\right) - \frac{8\pi e^2}{mk^2}\left|\bar{\Psi}\right|^2\right]
\end{aligned}
\tag{42}
$$

Both terms in the above equation diverge. The first term can be regulated using ultraviolet cut-off, which adds up to the background free energy. The second term can be treated by renormalizing the critical temperature of the superconducting transition with respect to its mean field value [5]. The free energy after renormalization becomes:

$$
F = F_n + a\left|\bar{\Psi}\right|^2 - \frac{16T\sqrt{\pi}e^3}{m^{3/2}}\left|\bar{\Psi}\right|^3 + \frac{b}{2}\left|\bar{\Psi}\right|^4 \tag{43}
$$

Notice that the third order term in the free energy can change the nature of the phase transition. Recall what is taught in class [4]; for the third order term with a negative sign, instead of a second-order phase transition when we lower the temperature, a *first order* phase transition will occur. Thus the fluctuation in magnetical field, although small in magnitude, changes the nature of the phase transition.

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