

Directed to Isotropic Percolation Transition in a 2-D Random Resistor-Diode Network

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In this work we study the random resistor-diode network on a 2-D lattice, which is a generalized percolation problem which contains both the directed and isotropic percolating phases. We use several methods in statistical mechanics with the objective of finding the phase diagram and the critical exponents. We first show by using mean-field arguments that the isotropic and directed percolating phases belong to different universality classes. Using the known thresholds for isotropic percolation and directed percolation on a square lattice and by employing symmetry arguments based on duality transformations, we determine all the fixed points, except for one. We use the position space renormalization group to find the remaining fixed point and to calculate the critical exponents of all the fixed points. Finally, we discuss the feasibility of observing this system experimentally.

I. INTRODUCTION

The percolation problem has been studied intensively for many decades, partly because it is a simple system to describe but with complex behavior and many physical realizations^{1,2}. The major difference between percolation and other phase transition models is the absence of a Hamiltonian. Instead, the theory is based entirely on probabilistic arguments and represents a classical geometric phase transition. In its original formulation, elements of a lattice (bonds or sites) are independently occupied with a probability p . At small p , finite clusters are formed and the probability that two atoms are connected decays exponentially with distance. This model has been applied to several problems such as magnetization of diluted magnets³ and formation of polymer gels⁴. More complicated models were developed to study bond-site percolation (both sites and bonds can be absent) and were used, for instance, to improve models of polymer gelation⁵. Additionally, a preferential direction for the bond was added to study directed percolation problems and was used to model transport in random systems with an external bias⁶, irreversible chemical reactions⁷, etc.

The isotropic and directed percolation problems can be generalized so that each bond can be directed in either or both directions (see Fig. 1 (a)). In this case, the system has two parameters, namely the bond concentration and the bond orientation, which can be adjusted independently to drive the system to a percolation threshold. Clearly, the system will exhibit richer phenomena than in the previous two percolation problems. In particular, we will have transitions between non-percolating, isotropically percolating and directed percolating phases. Additionally, the three phases meet in a line of multicritical points where both the bond concentration and bond orientation are relevant parameters.

The objective of this paper is to study the phases and critical behavior of the random resistor-diode network using different methods in statistical mechanics. First we will show that the isotropic percolating and directed percolating phases belong to two different universality classes by calculating the critical exponents using the saddle-point approximation and fluctuations. The differ-

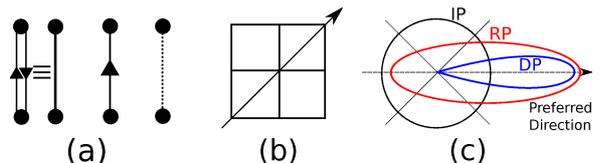


FIG. 1. Possible types of bonds in the random resistor-diode model. From left to right: resistor, diode and vacancy. (b) Preferred direction that determines positivity and negativity of diodes. (c) Typical cluster shape for the isotropic percolation (IS, $p_+ = p_- = 0$), directed percolation (DP, $p_+ \neq 0$, $p_- = 0$) and general case within the random resistor-diode model (RP).

ent nature between the phases is what makes the random resistor-diode network interesting. With knowledge of the fixed points of the isotropic percolation and directed percolation problems, and using symmetry arguments based on duality transformations, we can find all the fixed points except for one. To find the last fixed point, we use position-space renormalization group (PSRG) to find recursion relations for the parameters of the model. Using the same recursion relations, we find the critical exponents of this model. This problem was originally solved by Redner⁸ and Dhar *et al.*⁹ We will use mainly Redner's ideas with minor digressions to other work or additions from the author.

This paper is organized as follows: In Sec. II we describe the 2-D random resistor-diode model that will be used in this work. In Sec. III we describe the possible phases of the systems and show, using a Landau-Ginzburg free energy expansion, that isotropic and directed percolation belong each to different universality classes. In Sec. IV we find, using duality arguments, most of the fixed points in the model. In Sec. V we use PSRG to find an additional fixed point and calculate the critical exponents of this model. In Sec. VI we discuss possible experiments for percolation theory. Finally, in Sec. VII we summarize our work and give the conclusions.

II. RANDOM RESISTOR-DIODE MODEL

In the random resistor-diode model, each bond can be either vacant with probability q , or occupied by one of three possibilities: a resistor with probability p , a diode oriented in the positive direction with respect to a preferred axis with probability p_+ , and a diode oriented in the opposite direction with probability p_- (see Fig. 1 (a)). This model contains three independent parameters because of the relation $q + p + p_+ + p_- = 1$. To simplify the discussion, our lattice will be a square lattice with a preferential direction $\hat{n} = (1, 1)$ (see Fig. 1 (b)).

We see that this model reduces to the isotropic percolation problem when $p_{\pm} = 0$, and to the directed percolation problem when $p = p_- = 0$. It is important to notice that in isotropic percolation problem close to the percolation threshold, clusters have no particular shape, and thus, there is only one length of interest. However, for directed percolation, clusters have a preferred shape which is elongated in the preferred axis, and thus there are two relevant lengths (see Fig. 1 (c)). When we consider the random resistor-diode network, we can move from one phase to the other. In particular, there are regions in the phase diagram where the system is percolating isotropically and in a particular direction at the same time.

III. PHASES

From our previous discussion, we can argue that the system will present three different phases: a non-percolating phase (NP), an isotropically percolating phase (IP) and a directed percolating phase (DP; this phase actually occurs twice because directed percolation can occur in two directions, thus we have DP+ and DP-). In addition to the classical non-percolating to percolating phase transition, we will also have a new kind of transition in which the system is percolating isotropically and in a directed fashion at the same time when located in the multicritical line separating all phases.

Even though isotropic and directed percolation seem to be related problems, it can be shown that they actually belong to different universality classes. With this in mind, we can estimate the critical exponents for each model and also estimate the upper critical dimension to verify that this is indeed the case. A mean-field theory treatment can be derived for percolation problems using the Landau-Ginzburg expansion for the free energy. Considering that the isotropic percolation problem is the limit of the Potts model when $q \rightarrow 1$,¹⁰ we can use the Potts model free energy¹¹ in this limit

$$\mathcal{F}(P) = \frac{K}{2}(\nabla P)^2 + (p_c - p)P^2 + cP^3 + \dots \quad (1)$$

where p is the bond probability, p_c is its critical value, c is a positive constant and $P(x)$ is the order parameter which represents the percolation probability at point x

($P \geq 0$ and thus the cubic term provides the required stability). Within the saddle point approximation¹⁰, we can find the critical exponents β and γ . By considering a mean-field P_{sp} and minimizing Eq. (1) we find

$$P_{\text{sp}} = \begin{cases} 0 & , \text{if } p < p_c \\ 2(p - p_c)/3c & , \text{if } p > p_c \end{cases} \quad (2)$$

The exponent can be calculated directly from Eq. (2) and by considering $P \sim (p - p_c)^\beta$; thus, $\beta = 1$. To calculate γ we add the magnetic field contribution $-hP$ to the free energy of Eq. (1). Minimization within the saddle-point approximation now yields

$$h = 2(p_c - p)P + 3cP^2 \quad (3)$$

from which we obtain

$$\chi^{-1} = \left. \frac{\partial h}{\partial P} \right|_{h=0, P_{\text{sp}}} \propto P_{\text{sp}} \propto (p - p_c) \quad (4)$$

and therefore $\gamma = 1$. To study the critical exponent of the correlation length, we consider fluctuations in the order parameter $P = P_{\text{sp}} + \delta P$ and expand the free energy as

$$\mathcal{F} = \mathcal{F}_{\text{sp}} + \frac{K}{2}(\nabla \delta P)^2 + (p - p_c)\delta P^2 + o(\delta P^3) \quad (5)$$

when $p > p_c$. We notice that the free energy reduces to that of the gaussian model to lowest order and in consequence we obtain $\nu = 1/2$.

The directed percolation problem has a preferred axis and for this reason we need to include terms in the free energy that causes the anisotropy. The lowest order contribution of the anisotropy to the free energy is quadratic in order parameter and should contain one derivative in the preferred direction, i.e. $(\vec{n} \cdot \nabla P)P$. This addition does not affect β and γ , but it does affect the correlation length. The contribution of the fluctuations to the Hamiltonian in momentum space can be written as

$$\mathcal{H}_f \propto \int d^d \mathbf{q} \left\{ \left[(p - p_c) + \vec{n} \cdot \vec{q} + \frac{K}{2}q^2 \right] |\delta P_{\mathbf{q}}|^2 + o(\delta P^3) \right\}. \quad (6)$$

From Eq. (6) we see that the transverse fluctuations still have Gaussian behavior and therefore, the transverse correlation length ξ_{\perp} diverges as $\xi_{\perp}^{-1} \propto (p - p_c)^{\nu_{\perp}}$ with $\nu_{\perp} = 1/2$. However, for the longitudinal correlation length ξ_{\parallel} we have that the linear term dominates when $q \rightarrow 0$ and in consequence, $\xi_{\parallel}^{-1} \propto (p - p_c)^{\nu_{\parallel}}$ with $\nu_{\parallel} = 1$. This is consistent with our previous idea that the clusters shape will exhibit strong anisotropies as we reach the phase transition.

Using the Ginzburg criterion¹⁰

$$d_u \nu = 2\beta + \gamma \quad (7)$$

we can find the upper critical dimension d_u for the validity of the mean-field theory. For the isotropic percolation case the upper critical dimension is $d_u^{IP} = 6$. For the directed percolation case we need to modify the criteria to

take into account that clusters will diverge with the shape of a d -dimensional ellipsoid with one dimension equal to ξ_{\parallel} and the remaining $d - 1$ dimensions equal to ξ_{\perp} . For this case, the Ginzburg criterion is

$$(d_u - 1)\nu_{\perp} + \nu_{\parallel} = 2\beta + \gamma \quad (8)$$

and gives $d_u^{DP} = 5$. We see that the critical dimension is reduced compared to the isotropic percolation, which is analogous to the reduction of the critical dimension in systems with long-range interactions.

IV. SYMMETRIES OF THE PHASE DIAGRAM

Because the phase diagram contains four different parameters, it is best represented in the ‘composition’ tetrahedron shown in Fig. 2. Here we also add (with filled circles) the known values for the fixed points in isotropic and directed percolation. Because the square lattice is self-dual, the isotropic percolation value is exactly $p_c = 0.5$. However, for directed percolation the critical threshold has to be estimated numerically¹² and is approximately $p_c = 0.6447$ (in the next section we will estimate this value using PSRG). Additionally, we have the stable fixed points located in the vertices of the tetrahedron, making a total of 7 known fixed points (4 on vertices, 3 on binary lines).

New fixed points can be found by using symmetry considerations and the previously mentioned fixed points. The first symmetry is due to the fact that exchanging p_+ and p_- does not change the problem and thus, the phase diagram is symmetric in the $p_+ = p_-$ planes. Additionally, like in isotropic percolation, we can find another symmetry plane defined by $p = q$ by constructing the dual transformation.

The dual transformation can be done as follows: if the bond is a resistor, the dual is a vacancy and viceversa; if the bond is a diode, the dual is a diode rotated 90° clockwise (in this case we are also rotating the preferred axis; see Fig. 3 (a)).

There are a few cases to consider and which we describe in order of increasing complexity. In the first case, if the original lattice is formed only by finite clusters, then clearly the dual will be a lattice percolating isotropically because we have to fill with resistors the vacancies that separate each cluster. In the second case we assume that the original lattice contains a cluster that is percolating in one direction. When we construct the dual and rotate all the bonds, an infinite cluster is also formed. In principle it is not obvious that they will have the correct directional connectivity, but considering that positive bonds are transformed into positive bonds of the dual lattice, this is actually the case (see Fig. 3 (b) for an example). Finally, the most difficult case to visualize is the case in which an infinite cluster is not percolating because of wrong directionality of the diode bonds (see Fig. 3 (c)). The dual of this lattice is a lattice of disconnected clusters of resistors joined by the bonds where the

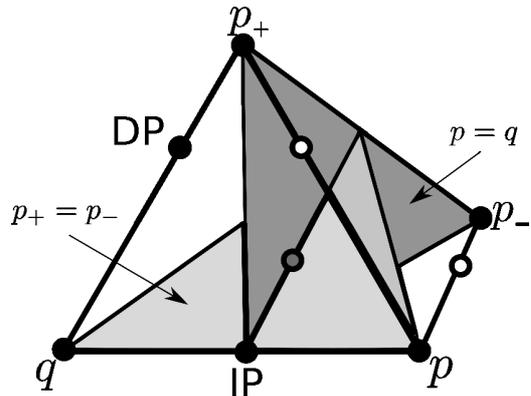


FIG. 2. Composition tetrahedron of the random resistor-diode model and the symmetry planes of the phase diagram. The IP and DP percolation thresholds are indicated in filled circles, while the fixed point obtained by symmetry are indicated in open circles. The circle filled in grey corresponds to the ‘mixed’ fixed point.

bond mismatch occurs. As shown in Fig. 3 (c), the mismatched bonds transform into a loop that can percolate in all directions joining all the disconnected clusters of resistors.

In conclusion, we argue that the positive and negative diode phases transform into themselves, while the vacancy and resistor phases transform into each other. Additionally, the square lattice transform into itself (i.e., its self-dual) and therefore, we conclude that there exists a symmetry in the $p = q$ plane. The intersection of both symmetry planes define a multicritical line where all phases meet. In particular, there will be a new isotropic threshold given by the expression $p_c = 1/2 - p_{\pm}$, which is a generalization of the isotropic percolation threshold.

We see from this analysis that we can find two new fixed points only on symmetry arguments from the 7 known fixed points (marked with open circles in Fig. 2), making a total of 9 fixed points. However, we are still missing one additional fixed point which is probably the most interesting one. Again, we can use symmetry consideration to determine that the line of intersection between the planes $p_+ = p_-$ and $p = q$ determine a multicritical line that separate the 4 phases and where this ‘mixed’ fixed point is located. This point cannot be determined by knowledge from the directed and isotropic percolation models and have to be determined by other means. In the next section we perform PSRG to determine the location of this fixed point and, additionally, to determine the stability of the remaining fixed point in order to complete the phase diagram.

V. POSITION SPACE RENORMALIZATION GROUP APPROACH

The PSRG approach is based on rescaling a $b \times b$ cell to a 1×1 cell and was first developed by Reynolds¹³ for

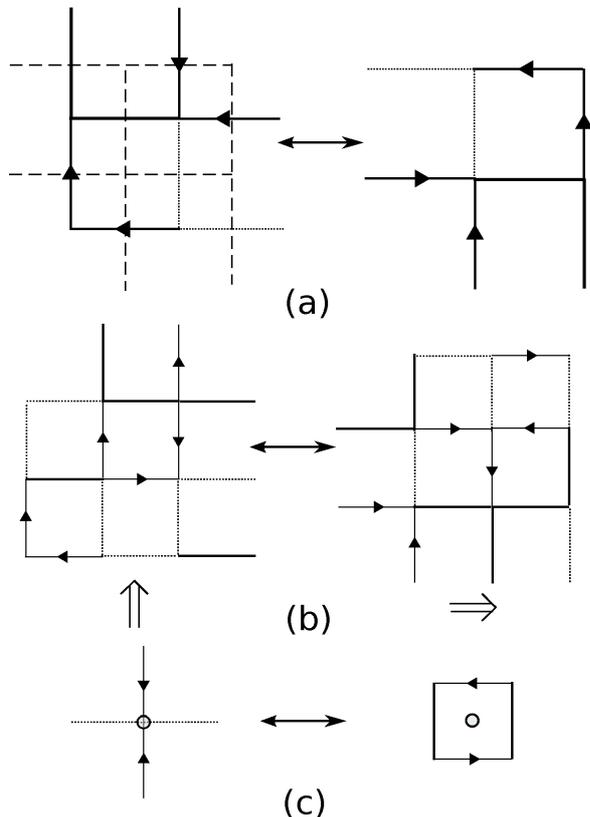


FIG. 3. (a) Duality transformation for the random resistor-diode network (b) Dual transformation of a directed percolating lattice. The dual is percolating in the same direction (indicated by an arrow) (c) Example of a mismatching bond. When we transform into the dual, the mismatching bond transforms into a loop that conducts in all directions.

the isotropic percolation problem and later generalized to directed percolation⁸. In the Reynolds's approach, a weight function is used in which percolating cells map to an occupied bond. In our case, we need to extend this idea to include directed percolation. This can easily be done⁸ by considering that each cell can map either to a diode, a resistor or a vacancy. Here we will work with the case $b = 2$ to find the recursion relations for the parameters p' , p'_+ , p'_- and q' .

In Fig. 4 (a) we plot the initial cell (containing 8 bonds) and which, if repeated periodically, will form the square lattice. The assignments are as follows: if the cell can percolate in both directions (up and down) then we replace the cell by a resistor; if it can percolate in one direction then we replace the cell by a diode in the same direction; and finally, if the cell does not percolate we replace the cell by a vacancy. From these assignments we see that, in reality, only 5 of the 8 bonds determine if the cell percolate or not (see Fig. 4 (a)) and thus, there are in total 4^5 possibilities. In Fig. 4 (b) we show a few examples of the renormalization. This problem was originally solved by Redner⁸ who first found the corresponding recursion relations and which can be found in the

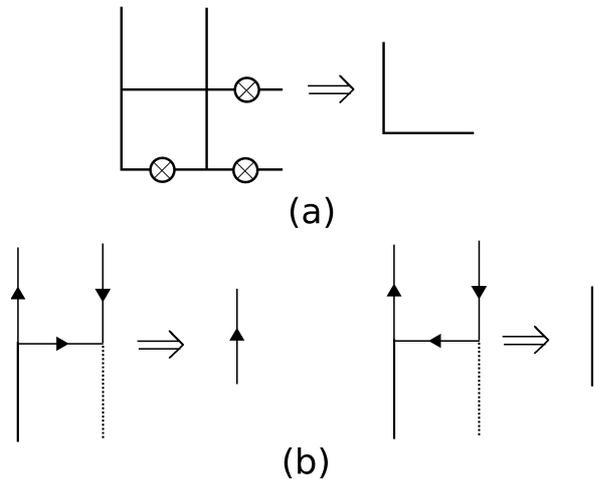


FIG. 4. (a) Mapping of a 2×2 cell into a 1×1 cell. In the 2×2 cell it is indicated with crossed circles the bonds that do not contribute to cell percolation. (b) Examples of mappings that generalize the ones performed by Reynolds¹³.

appendix. These recursion relations represent a generalization of the Reynolds recursion relations for isotropic percolation, and the latter can be obtained by considering the case $p_{\pm} = 0$. Here we see that we make the PSRG assignments on the up-down direction of percolation and therefore, the PSRG is sensitive to the longitudinal critical exponents. This method is not valid on the transverse direction.

A. Mixed Fixed Point

To find the missing fixed point we evaluate the recursion relation at the intersection of the symmetry planes $p_+ = p_-$ and $p = q$. The recursion relation for p'_+ in this line is given by

$$p'_+ = 10p_+^5 + 50pp_+^4 + 92p^2p_+^3 + 76p^3p_+^2 + 26p^4p_+ \quad (9)$$

where $p = 0.5 - p_+$. From the derivative of Eq. (9) at $p_+ \approx 0$ we see that the isotropic percolation fixed point ($p_{\pm} = 0$ and $p = q = 0.5$) is unstable in this line and therefore, we can iterate (using a hand-calculator) Eq. (9) to find the 'mixed' fixed point (see Fig. 5 (a)). We find that this point is located at $p_+^* = p_-^* = 0.2457$ and $p^* = q^* = 0.2543$. Now we have all the fixed point of the random resistor-diode network and we can proceed to study the stability of all points using the PSRG recursion relations.

B. Stability and Relevant Parameters

We use the PSRG equations to find the stability of the 'mixed' fixed point only, and the same procedure can be used to find the stability of the remaining fixed points

p_+^*	p_-^*	p^*	q^*	Class.	Eig.	ν_{\parallel}
0	0	1/2	1/2	IP	1.625($\times 2$)	1.428($\times 2$)
					1.5	1.710
0.5550	0	0	0.445	DP($\times 2$)	1.567	1.543
0.5550	0	0.445	0	DPS($\times 2$)	1.567	1.543
0.2457	0.2457	0.2543	0.2543	Mixed	1.625	1.428
					1.5	1.710

TABLE I. Location, relevant eigenvalues and critical exponent ν_{\parallel} of the fixed points of the random resistor-diode network. Trivial fixed points (located in the vertices of the tetrahedron) are excluded from the table. The number of relevant directions can be deduced from the number of relevant eigenvalues. The points classified as DPS correspond to the symmetric points with respect to the plane $p = q$ and indicated with open circles in Fig. 2.

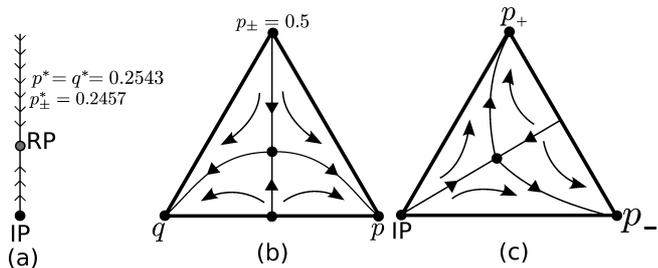


FIG. 5. (a) PSRG flow on the high-symmetry line using Eq. (9) (b) PSRG flow on the symmetry plane $p_+ = p_-$ (c) PSRG flow on the symmetry plane $p = q$.

which we will only tabulate. Linearizing the equations of appendix A around the fixed point we obtain

$$\mathcal{L}(p^*, p_{\pm}^*) = \begin{pmatrix} 1.625 & 0.537 & 0.537 \\ 0 & 1.025 & -0.475 \\ 0 & -0.475 & 1.025 \end{pmatrix} \quad (10)$$

where the components of the matrix are defined as $\mathcal{L}_{ij} = \partial_i \alpha' / \partial_j \beta$ and $\alpha, \beta = p, p_+, p_-$. The eigenvalues are $\lambda = 1.625, 1.5, 0.550$ with directions $(1, 0, 0)$, $(0, 1, -1)$ and $(-1, 1, 1)$, respectively. Therefore, we have two relevant directions, one contained in the $p_+ = p_-$ plane (and perpendicular to the one discussed in Fig. 5 (a)), and the other contained in the self-dual plane $p = q$. A similar approach can be applied for the other fixed points, and the results are summarized in Table I.

We are now ready to draw the phase diagram and PSRG flows. In Fig. 5 (b) and (c) we plot the renormalization group flows corresponding to the ‘mixed’ fixed point in the symmetry planes $p_+ = p_-$ and $p = q$, respectively. The complete phase diagram on the tetrahedron is shown in Fig. 6 (a) where all fixed points are indicated with dots and flows with arrows. The projection of the phase diagram on the faces of the tetrahedron are shown in Fig. 6 (b), where all the possible phases are also indicated.

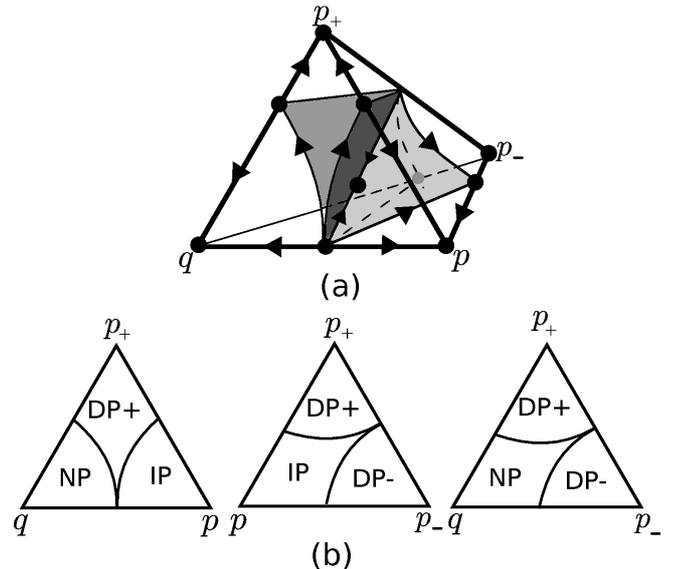


FIG. 6. (a) Phase diagram in the composition tetrahedron. All the fixed points are indicated with filled dots and the PSRG flows with an arrow. (b) Phases of the model on three of the faces of the tetrahedron.

VI. EXPERIMENTS ON PERCOLATION

Most of the work done related to percolation is theoretical and numerical. This is mainly due to the fact that the model is simple enough to construct, yet contains complex behavior and can require advanced techniques. On the experimental side, even though there are a number of known systems that do percolate, it becomes a hard task to control the parameters with sufficient precision in order to measure the critical exponents and verify the theory. Although isotropic and directed percolation in its simplest form contain only one relevant parameter, they both have represented an experimental challenge. The fate seems to be even more discouraging for the random resistor diode, where it is required to control three parameters simultaneously and with high precision. For instance, if we consider the flow of a fluid in a porous structure, it is difficult to fabricate materials with uniform porosity and to know with precision its value. Moreover, the need to make a large number of samples to obtain statistically meaningful results makes the experiments even more discouraging.

Early experiments were done in relation with isotropic percolation¹⁴ in which critical exponents could be measured successfully. However, only very recently were critical exponents measured for directed percolation in liquid crystals^{15,16} which are in good agreement with the theory. Achieving both IP and DP phases in a controlled manner at the same time seems to be an unattainable task for the time being.

VII. CONCLUSION

In this work we studied the 2-D random resistor-diode network. The system presents three phases which are the non-percolating phase, an isotropically percolating phase and a directed percolating phase. The model contains three parameters, and for specific conditions the system can be percolating both directed and isotropically. We showed with simple arguments that the isotropic and directed percolating phase belong to different universality classes. Using the known thresholds for isotropic and directed percolation and using symmetry arguments based on duality transformations, we determined all except one

fixed point. The remaining fixed point is the most interesting one and is located in the multicritical line. We determined the approximate location of this point by means of the position space renormalization group. At this point, there are only two relevant directions. We also used the position space renormalization group to calculate all the critical exponents. Finally, we commented on the difficulty of observing this particular system experimentally.

Appendix A: Position Space Renormalization Group

Recursion relations for the renormalized parameters p' , p'_+ , p'_- and q' of the random resistor-diode network.

$$p'_+ = 2p^3p_+^2 + p^2(7p_+^3 + 7p_+^2p_-) + p(5p_+^4 + 14p_+^3p_- + 6p_+^2p_-^2) + (p_+^5 + 5p_+^4p_- + 4p_+^3p_-^2) + q \{ p^3(4p_+) + p^2(22p_+^2 + 14p_+p_-) + p(20p_+^3 + 36p_+^2p_- + 8p_+p_-^2) + (5p_+^4 + 14p_+^3p_- + 6p_+^2p_-^2) \} + q^2 \{ p^2(17p_+ + p_-) + p(22p_+^2 + 14p_+p_-) + (7p_+^3 + 7p_+^2p_-) \} + q^3(4pp_+ + 2p_+^2) = f(p_+, p_-, p, q) \quad (\text{A.1})$$

$$p'_+ = p^5 + p^4(5p_+ + 5p_-) + p^3(8p_+^2 + 20p_+p_- + 8p_-^2) + p^2(3p_+^3 + 23p_+^2p_- + 23p_+p_-^2 + 3p_-^3) + p(6p_+^3p_- + 16p_+^2p_-^2 + 6p_+p_-^3) + (3p_+^3p_-^2 + 3p_+^2p_-^3) + q \{ 5p^4 + p^3(16p_+ + 16p_-) + p^2(8p_+^2 + 32p_+p_- + 8p_-^2) + p(8p_+^2p_- + 8p_+p_-^2) + 2p_+^2p_-^2 \} + q^2 \{ 8p^3 + p^2(6p_+ + 6p_-) \} + 2q^3p^2 = g(p_+, p_-, p, q) \quad (\text{A.2})$$

$$p'_- = f(p_-, p_+, p, q), \quad (\text{A.3})$$

$$q' = g(p_+, p_-, q, p). \quad (\text{A.4})$$

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