Self-Organized Criticality in Two Simple Lattice Models

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The concept of self-organized criticality was proposed in 1987 to explain the natural occurrence of self-similarity, in particular fractals. Similar to critical behaviors near a phase transition, self-organized criticality exhibits a high degree of self-similarity. However, it does not require the fine-tuning of a specific parameter (such as the temperature or the pressure). Criticality is spontaneously achieved in a self-organized manner. I will review two simple lattice models—the Bak-Tang-Wiesenfeld sandpile mode and the spring-block model for earthquakes—as well as related analytical and experimental results.

I. INTRODUCTION

Critical behaviors near a phase transition are one of the most remarkable phenomena in statistical mechanics. When a specific parameter of an equilibrium system—usually the temperature or the pressure—approaches a critical value, the system may undergo a phase transition from an ordered phase to a disordered one. Near the critical point, many statistical properties of the system exhibit singular behaviors, which are called "critical behaviors." In particular, the correlation length diverges to infinity at the critical point. In other words, if we rescale the coordinates ("zoom out") and conduct coarse-graining, the renormalized system will resemble its original form. Therefore, the system shows a high degree of self-similarity near its phase transition.

The phenomenon of self-similarity is also frequently observed in nature, and it usually occurs in the form of "fractals." For example, tiny streams of water join and form a larger stream in river networks. If we zoom out, we will find these larger streams forming creeks, and creeks forming rivers. Therefore, a river network resembles itself on drastically different scales. A similar property can be found in many other natural systems, including turbulent vortices, mountain landscapes, forest fires, and earthquakes. They all exhibit a surprising extent of self-similarity.

However, it is unlikely that these natural self-similarities can be explained by the traditional critical behaviors near a phase transitions. First of all, a phase transition requires exact fine-tuning of the critical parameter. For example, critical behaviors of the Ising model only manifest when the coupling constant K approaches its critical value $K_c = 1/2 \cdot \ln(\sqrt{2}+1)$. However, parameters of a natural system (such as the temperature and the pressure) are almost always changing. Therefore, it is unlikely that a natural system always stays near its critical point for the phase transition. Secondly, natural systems are rarely in equilibrium. They are constantly influenced by both dissipative forces and inputs of energy. Therefore, equilibrium statistical mechanics is not suitable for natural systems.

To overcome these difficulties, Bak, Tang, and Wiesenfeld jointly proposed the idea of "self-organized criticality" (abbreviated as SOC) in 1987 [1]. In particular, they show that it is possible for a large dissipative system to spontaneously achieve a critical state and stay there despite small perturbations. This kind of "criticality" is similar to traditional phase transitions becasue they both exhibit self-similarity—the system does not have any characteristic length scales. However, SOC fundamentally differs from traditional critical behaviors in two ways. On one hand, SOC does not require fine-tuning of parameters and is thus "self-organized." Therefore, SOC provides a more reasonable description of natural phenomena, because in a natural process physical conditions are constantly changing. On the other hand, SOC refers to a dynamic *steady* state instead of an equilibrium state. Unlikely equilibrium statistical mechanics, SOC describes both the spatial and the temporal evolution of a system. The critical state is thus a constantly changing and yet stable configuration under the continuous influence of dissipation and an input of energy.

In this paper, I will review two simple lattice models that exhibit SOC. The first one—the Bak-Tang-Wiesenfeld (abbreviated as BTW) sandpile model [1]—is a "discrete cellular automaton" (a term referring to integers on a lattice that are updated according to their neighbors) that describes the addition and the avalanche of sand on a plate. Remarkably, the BTW model (and its generalization) is partly solvable because the operations on sand nicely form an Abelian group [2]. The second one—the spring-block model for earthquakes [3]—is a continuous cellular automaton (real numbers instead of integers) that describes the accumulation of energy under the tectonic plate and its release in the form of earthquakes. Although less mathematically tractable, the spring-block model matches experimental observations and furthers shows that, unlike traditional phase transitions, SOC does not have universal critical exponents. By robustly reaching criticality in a self-organized manner, these models of SOC offer a possible explanation to the prevalence of self-similarity in natural systems.

II. THE BTW SANDPILE MODEL

A. Original Proposal

The BTW sandpile model was proposed by Bak, Tang, and Wiesenfeld as the first model that exhibits SOC [1]. On a two-dimensional square lattice of size $L \times L$, each site (x, y) is assigned an integer $z(x, y)$ representing the height of sandpile at this site. If the height of a site exceeds a critical level, defined as z_c, this site will become unstable. In this case, the site *topples*, and four grains of its sand relocate to its four nearest neighbors. In other words, the rules for updating this cellular automaton can be written as

$$\begin{aligned} z(x, y) &\to z(x, y) - 4 \\ z(x \pm 1, y) &\to z(x \pm 1, y) + 1 \\ z(x, y \pm 1) &\to z(x, y \pm 1) + 1 \end{aligned} \tag{1}$$

if $z(x, y) > z_c$. As we will see later, the *order* in which we topple each site does not matter. Interestingly, this model is non-parametric. The maximal height z_c is not a parameter because the system is invariant after adding a constant integer to every $z(x, y)$. Therefore, the choice of z_c does not affect the dynamics of the system, and it is commonly chosen that $z_c = 4$.

The boundary condition is chosen so that all boundary sites have zero heights. They thus resemble the edges of a plate of sand. Whenever a site topples near the boundary, some excessive grains of sand will fall off the plate (*i.e.* dissipation). Therefore, by toppling every site that has an unstable height $z(x, y) > z_c$ (in an arbitrary order as we shall see later), we can *relax* the system to a stable state in which every height $z(x, y) \leq z_c$.

B. Simulation Results

Bak, Tang, and Wiesenfeld simulated this model on a $L = 50$ or 100 square lattice [1]. The system is initialized by randomly and independently assigning a large integer $z(x, y) >> z_c$ to each site and then relaxing the system. This stable state is then perturbed by adding one grain of sand at a time to a random site until a toppling is triggered. Toppling of the perturbed site may induce further toppling of nearby sites, generating an *avalanche*. The upper panel of Fig. 1 shows the size of some possible avalanches (marked by black dots) that can be triggered in a stable state ($L = 100$).

After the relaxation of each avalanche, the system will again reach a (probably different) stable state. This new state can again be perturbed by random additions of sand until a new avalanche occurs. Therefore, instead of an equilibrium state, the system will reach a dynamic *steady* state under the continuous input of sand and the sudden occurrences of avalanches. The lower panel of Fig. 1 shows the simulated probability distribution of the size of these avalanches, defined as the number of sites that are affected ($L = 50$).

From the size distribution of avalanches (Fig. 1) we see a striking degree of self-similarity. Avalanches can occur on drastically different length scales: from a tiny avalanche that affects only one site, to a huge one that affects the entire plate. Therefore, the size s of avalanches follows a "scale-free" distribution (which means the absence of any characteristic length scale) in the form of a power law:

$$Prob(s) \sim s^{-\tau} \tag{2}$$

where $\tau \approx 0.98$. Scale-free power law distributions are characteristic of fractals. A similar result is also observed in the three-dimensional case, but the critical exponent is different ($\tau \approx 1.35$).

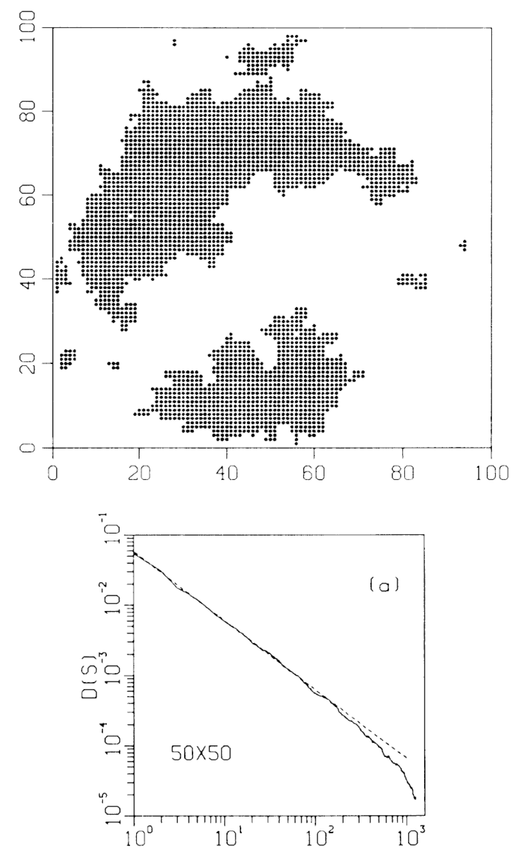

FIG. 1: Simulations of the BTW sandpile model, adopted from [1]. The upper panel shows the possible avalanches (black dots) that can be triggered in a stable state ($L = 100$). The lower panel shows the distribution of avalanche sizes (the number of affected sites) under perturbations ($L = 50$).

C. Generalization to an Abelian Group

Dhar found that the BTW model is partly solvable because it resembles an Abelian group [2]. The original BTW model is both undirected (a site influences a neighboring site in the same manner as this neighbor influences it) and unweighted (every site follows the same rule). Dhar generalized this model to an arbitrary graph that is both directed and weighted. In particular, neighbors of a site i are updated as follows:

$$z_j \to z_j - \Delta_{ij} \tag{3}$$

if $z_i > z_{i,c}$. Similarly, the dynamics does not depend on the choice of $z_{i,c}$. For convenience we can pick $z_{i,c} = \Delta_{ij}$.

In this model, the matrix Δ_{ij} must satisfy the following criteria: (1) Each element Δ_{ij} is positive if $i = j$, and non-positive if $i \neq j$. This ensures that toppling is a diffusive process, in which sand flows from an unstable site to its neighbors. This is also required for the Abelian properties. (2) $\Sigma_j \Delta_{ij} \geq 0$ so that toppling does not generate more sand. (3) Furthermore, there exists at least one site i with $\Sigma_j \Delta_{ij} > 0$, and all other sites should have a path towards one of these sites. Toppling of these sites decreases the total grains of sand. They are thus dissipative and correspond to the boundaries in the BTW model.

Given a stable configuration $\mathcal{C}$, we can define an operator a_i so that $a_i\mathcal{C}$ is the configuration after adding one grain of sand to site i and relaxing the system (if necessary). Dahr rigorously proved that these operators commute (the Abelian property), namely

$$a_i a_j = a_j a_i \tag{4}$$

for all i and j. The intuition is that when the system has two unstable states i and j to relax, it does not matter which one to topple first because the net change in any site k will always be $-\Delta_{ik} - \Delta_{jk}$. Furthermore, it is also equivalent whether we topple an unstable state i *before* or *after* adding a grain of sand to site j.

Another property of these operators a_i is that adding Δ_{ii} grains of sand to site i is equivalent to adding Δ_{ij} grains of sand to all other sites $j \neq i$, because they only differ by a toppling action. Therefore,

$$a_i^{\Delta_{ii}} = \Pi_{j \neq i} a_j^{-\Delta_{ij}} \tag{5}$$

for all i. Therefore, we can always reduce the power of a_i to below Δ_{ii}. These commuting operators a_i thus form a finite Abelian semi-group.

Now we need to classify all possible configurations $\mathcal{C}$ into two categories—transient or recurrent. Because the number of configurations is finite in this system, continuously acting a_i on a configuration will eventually lead to a limit cycle. In other words, $a_i^{n+p}\mathcal{C} = a_i^n\mathcal{C}$ for a period p. Therefore, with respect to a_i, each configuration must be either on a repeated cycle (*i.e.* recurrent) or not (*i.e.* transient). Interestingly, a recurrent configuration for a_i is also a recurrent configuration for a_j because of the Abelian property ($a_i a_j^p = a_j^p a_i$). Furthermore, it can be proved that all recurrent configurations can be reached from each other via some combination of operations a_i.

If we consider only these recurrent configurations, they will form an Abelian group generated by operators a_i. We can further define the inverse operator a_i^{-1} as taking one step back on the limit cycle. In this way, Eq. (5) can be rewritten to define the identity operator

$$\Pi_j a_j^{\Delta_{ij}} = 1. \tag{6}$$

We are now ready to solve the steady state under time evolution. If each site is perturbed (by adding a grain of sand) with a probability p_i, the time evolution operator would then be $\mathcal{W} = \Sigma_i p_i a_i$. $\mathcal{W}$ can be diagonalized by diagonalizing all a_i simultaneously because they commute with each other. With the help of Eq. (7), Dahr found that the steady state of the system is *all recurrent configurations occurring with exactly the same probability*, regardless of the values of p_i. He further proved that the number of these recurrent configurations equals to $\det(\Delta)$.

D. Analytical Results

Dahr's generalization to an Abelian group offers some analytical results for the BTW model. The two-dimensional BTW model corresponds to $\Delta_{ii} = 4$ and $\Delta_{ij} = -1$ for nearest neighbors $j \neq i$. The number of recurrent configurations can be calculated by diagonalizing Δ via Fourier modes:

$$\det(\Delta) = \Pi_{l=1}^{L} \Pi_{m=1}^{L} [4 - 2\cos(2\pi l/(L+1)) - 2\cos(2\pi m/(L+1))] \tag{7}$$

which approaches $(3.210...)^N$ ($N = L^2$ is the number of sites) when $L \to \infty$. This is exponentially smaller than the total number of configurations 4^N. Therefore, at the steady state, only a small fraction of all possible configurations can occur. Dahr also found an analytical formula for the probability distribution of the height at each site: $Prob(z_i) = 0.0736..., 0.1739..., 0.3063..., 0.4461...$ for $z_i = 1, 2, 3, 4$ respectively.

However, little is known about the critical exponents in two dimension. By an equivalence to the Potts model with $q \to 0$ and to spanning trees, Dahr found that if avalanches are relaxed in a wave-by-wave fashion, the size of the last wave should follow a power law distribution $\sim s^{-\tau_w}$, where $\tau_w = 11/8$. The exponent τ for the size of avalanches should be at least smaller than this value, but we do not have any analytical result yet.

Dahr's analysis also yields interesting conclusions about different dimensions. In one dimension (a chain), although the number of possible configurations (2^L) increases exponentially with L, the number of recurrent ones increases only linearly ($\det(\Delta) = L + 1$). These $L + 1$ configurations can be found explicitly, and perhaps surprisingly, they do not exhibit critical behaviors. In particular, as $L \to \infty$, an avalanche almost always sweeps an infinite number of sites. In three dimensions (cubic lattice), the critical component is analytically calculated as $\tau = 4/3$, which is consistent with Bak, Tang, and Wiesenfeld's simulations ($\tau \approx 1.35$). There is no need to go to higher dimensions, because the upper critical dimension can be shown to be 4 by an equivalence to loop-erased random walks.

III. THE SPRING-BLOCK MODEL FOR EARTHQUAKES

A. Formulation

In 1992, Olami, Feder, and Christensen proposed another model that exhibits SOC, and used it to explain the power law distribution of earthquake energies [3]. As shown in the upper panel of Fig. 2, this model describes the movement of blocks on a two-dimensional square lattice between two large plates. Each block (x, y) is connected to its four nearest neighbors and a large moving plate via springs. The spring constant is K_1 in the left-right direction, K_2 in the front-back direction, and K_L if it connects to the moving plate.

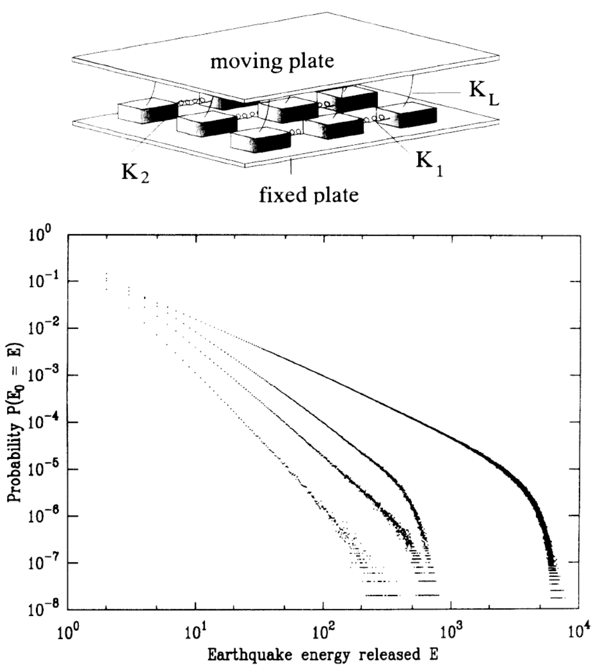

FIG. 2: Simulations of the spring-block model for earthquakes, adopted from [3]. The upper panel shows the simulation scheme, in which blocks on a square lattice are connected to each other and a moving plate via springs. The lower panel shows the distribution of earthquake energies (the number of slipping) under perturbations (lattice size 35×35). From left to right, the parameter $\alpha = 0.10, 0.15, 0.20, 0.25$.

This spring-block system evolves when the upper plate moves horizontally with a constant velocity. If the blocks stay at rest, the net force on each block will increase at the same rate that is proportional to K_L. If the net force on a block $F(x, y)$ exceeds a critical value F_c, this block will slip and reduces its force to zero. Such slipping changes the force on nearby blocks, and the general rule can be shown as follows:

$$\begin{aligned} &F(x,y) \to 0 \\ &F(x \pm 1, y) \to F(x \pm 1, y) + \alpha_1 F(x,y) \\ &F(x, y \pm 1) \to F(x, y \pm 1) + \alpha_2 F(x,y) \end{aligned} \tag{8}$$

if $F(x, y) > F_c$. The constants are $\alpha_1 = K_1/(2K_1 + 2K_2 + K_L), \alpha_2 = K_2/(2K_1 + 2K_2 + K_L)$. Here we will concentrate on the isotropic case, in which $K_1 = K_2$ and $\alpha_1 = \alpha_2 = \alpha$.

Although closely related to the BTW model, the two models are different in several aspects. First of all, the BTW model is a discrete cellular automaton (with integers), whereas the spring-block model is a continuous one (with real numbers). Secondly, during relaxation the BTW model always transfers a fixed amount of sand (4 on a square lattice), whereas the spring-block model transfers an variable amount of force depending on the state of the unstable block. Last but not least, toppling in the BTW model conserves the total number of sand, whereas slipping in the spring-block model changes the sum of all forces (unless $K_L = 0$). These properties create significant difficulties in analytically solving this system, and Dahr's Abelian method does not work for this earthquake model.

B. Simulation Results

Olami, Feder, and Christensen simulated the spring-block model on a 35×35 lattice. Similar to the steady state in the BTW model, the system experiences a constant input of energy (the movement of the upper plate) and sudden occurrence of relaxations—the "earthquakes." The lower panel of Fig. 2 shows the distribution of earthquake energies, defined as the total number of slipping, with different values of $\alpha = 0.10, 0.15, 0.20, 0.25$. Again the system exhibits self-similarity in the form of a power law:

$$Prob(E) \sim E^{-\tau} \tag{9}$$

if the parameter α is in the range $0.05 \sim 0.25$. Interestingly, the absolute value of the critical exponent τ (steepness of the curve) decreases when α increases. Therefore, in SOC the critical exponent does depend on the details of a model. This is different from traditional phase transitions, in which critical behaviors fall into universality classes.

C. Comparison with Observations

SOC in the spring-block model is consistent with the observed distribution of earthquake energies. In fact, the famous Gutenberg-Richter law states that regardless of the scale of earthquakes, earthquake energies follow a power law distribution with an exponent $\tau = 1.80 \sim 2.05$. This agrees with the spring-block model if the parameter $\alpha = 0.18 \sim 0.22$, which is reasonable if all spring constants are approximately equal ($K_1 \approx K_2 \approx K_L$, yielding $\alpha \approx 0.20$). Therefore, the spring-block model offers a quantitative explanation to the natural self-similarity of earthquakes.

IV. CONCLUSION

In this paper, I reviewed the concept of SOC in two simple lattice models—the BTW sandpile model and the spring-block model for earthquakes. First proposed in 1987, SOC is a statistical phenomenon closely related to critical behaviors in traditional phase transitions. In particular, SOC exhibits self-similarity on drastically different length scales. However, unlike phase transitions, SOC does not require fine-tuning of a specific parameter (such as the temperature or the pressure). The system spontaneously achieves a critical state in a self-organized manner. In addition to the two simple models in this paper, many other families of SOC models have been discovered. For example, the forest fire model explains the observed power law distribution of the size of forest fires, although the critical exponent does not perfectly match [4]. In conclusion, SOC provides important insights into the frequent occurrence of self-similarity (fractals) in nature.

Unfortunately, the study of SOC has encountered significant difficulties. Although SOC models are usually mathematically simple, they turns out to be difficult to solve analytically. In particular, the study of SOC lacks a systematic framework as powerful as the renormalization groups in statistical mechanics. It is also unclear what components of the models are crucial to the emergence SOC. For example, it has been debated whether SOC requires an infinitely slowly input of energy.

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