# From $d$-dimensional Quantum to $d+1$-dimensional Classical Systems

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I review the mapping from the partition function of a $d$-dimensional quantum system to that of a $d+1$-dimensional classical system and I apply it to several models, in particular the classical $1d$ Ising, $2d$ XY, and $3d$ Ising models. These examples respectively illustrate the general features of the classical/quantum correspondence, a Berry phase term, and the role of gauge constraints. The limitations of the mapping due to the Berry phase term from the quantum side are discussed in the context of the Heisenberg spin chain.

Dualities between two physical models serve as an important analytical tool whereby intractable or obscure aspects of one model can be elucidated by more manifest aspects of the other model. For example, the holographic principle [1], in which a bulk gravitational theory of a space-time is mapped to a conformal field theory on the boundary of the space-time, has served as a valuable dictionary for both string theorists and condensed matter physicists to extract information of strongly-coupled systems from weakly-coupled gravitational systems. The object of this paper is more down to earth; here I review a fairly general procedure allowing one to map the partition function of a $d$-dimensional quantum system to the partition function of a $d+1$-dimensional classical system. This procedure, known as the quantum to classical (QC) mapping, can be used to gain insight into quantum phase transitions from their classical counterparts or understand new quantum universality classes from Monte Carlo simulations of the appropriate classical $d+1$-dimensional systems [2].

The structure of this paper is as follows. First, I introduce the key ideas behind the quantum to classical mapping and claim several general features of the mapping. These are illustrated explicitly in the next section, in which the $0d$ (quantum) and $1d$ (classical) Ising models are mapped to each other. Next, I apply the mapping to the classical 2d XY model and 3d Ising models, and I note how the duality within the latter model maps to a duality within the corresponding quantum model. Finally, I briefly mention further successes of the mapping as well as systems where it fails due to Berry phases. I conclude with a short recap and some harmless speculation.

## QUANTUM TO CLASSICAL MAPPING

The main idea behind the quantum to classical mapping is that, for a $d$-dimensional quantum system with Hamiltonian $H_q$ and at temperature $1/\beta$, the quantum partition function $Z = tr(e^{-\beta H_q})$ can be evaluated using the imaginary time path integral:

$$Z = \sum_x \sum_{x_1,...x_N} \langle x|e^{-H_q\delta\tau}|x_1\rangle\langle x_1|...|x_n\rangle\langle x_N|e^{-H_q\delta\tau}|x\rangle$$

where $\delta\tau$ is much smaller than all time scales of $H_q$ and $N\delta\tau = \beta$. The $x$'s are intermediate states.

However, this expression is precisely the partition function of a classical system in $d+1$ dimensions (the quantum imaginary time is a classical spatial dimension) evaluated using the transfer matrix technique with transfer matrix

$$T = e^{-H_q\delta\tau} \approx 1 - H_q\delta\tau. \tag{1}$$

The classical system has $\beta/\delta\tau$ sites in the $(d+1)$th dimension; hence, a $T=0$ quantum system corresponds to a classical system with infinitely many sites.

In addition, this quantum to classical mapping has several more general features. For example, as evident in the expression for $T$ above, the temperature of the classical system is determined by the coupling constants of $H_q$. The most nontrivial correspondence is the inverse relation between the correlation length $\xi_c$ of the classical system in the $(d+1)$th dimension and the characteristic energy scale of the quantum system (for example, the energy gap between the ground and first excited state) [2]. Finally, note that the reverse procedure of mapping a classical system to a quantum system must be done carefully so that $H_q\delta\tau$ is small; it turns out that this only works in what is called the 'scaling limit' [3] of the classical system: when $\xi_c \gg a$, the lattice spacing. In this limit, the microscopic details of the classical model are not relevant, and it makes sense that only in this scaling limit should the 'universal' aspects be related to a quantum model.

## APPLICATION TO CLASSICAL $1d$ ISING MODEL

I will now illustrate all these features by mapping the $1d$ classical Ising chain in the scaling limit to a $0d$ quantum Hamiltonian.

For simplicity, I consider the classical Ising chain with no magnetic field but with on-site energies:

$$\beta H_c = -K \sum_{<ij>} (s_i s_j - 1), \tag{2}$$

where the $c$ subscript denotes classical, the brackets denote nearest neighbors, and $s = \pm 1$. I will also assume the chain has periodic boundary conditions.

As is well known [4], one can attain the partition function exactly by using the transfer matrix technique. For the Hamiltonian considered above, the transfer matrix is

$$T = \begin{pmatrix} 1 & e^{-2K} \\ e^{-2K} & 1 \end{pmatrix} = I + e^{-2K}\sigma_x, \tag{3}$$

where $I$ is the identity matrix and $\sigma$ are the Pauli matrices. For a chain with $N$ sites, the partition function is

$$Z = tr(T^N) = (1 - e^{-2K})^N + (1 + e^{-2K})^N. \tag{4}$$

The correlation function $\langle s_l s_m \rangle$, found by inserting $\sigma_z$ at sites $l$ and $m$ in the above trace, reduces to $\langle s_a s_b \rangle = (\tanh K)^{m-l}$ in the $N \to \infty$ limit. This yields the correlation length

$$\xi = -a \ln \tanh K \approx \frac{a e^{2K}}{2} \tag{5}$$

for $K \gg 1$. (Here $a$ is the lattice spacing).

I now seek to find the $0d$ quantum Hamiltonian that has the same partition function (4) as the classical chain. From expressions (1) and (3), it is evident that

$$H_q \delta\tau = -e^{-2K}\sigma_x. \tag{6}$$

Only in the scaling limit $e^{2K} = \frac{\xi}{a} \gg 1$ is $H_q \delta\tau$ small so that the path integral subdivision holds. Hence, the classical Ising chain in the scaling limit has the same partition function as the quantum system with Hamiltonian

$$H_q = -\frac{\Delta}{2}\sigma_x \tag{7}$$

at inverse temperature $\beta_q = \frac{Ne^{-2K}}{\Delta/2} = \frac{Na}{\Delta\xi}$; $\Delta$ is a free parameter. As advertised near the beginning of this section, the classical system size is proportional to the quantum inverse temperature, and the quantum energy gap and classical correlation length are inversely related.

## APPLICATION TO CLASSICAL $2d$ XY MODEL

As a more interesting application of the duality, I now show how the $1d$ quantum XY model maps to the $2d$ classical XY model for some parameters [2, 5]. The cited works only considered a given parameter (0) for the mean rotor angular momentum, but there is some virtue in generalizing it to be an arbitrary integer: one encounters a Berry phase term which in this situation is innocuous but in other situations makes the QC mapping less useful.

Consider the quantum $1d$ XY model

$$H_q = U \sum_i (n_i - \bar{n})^2 - t \sum_{<ij>} \cos(\theta_i - \theta_j), \tag{8}$$

where $n_i = -i\frac{\partial}{\partial \theta_i}$ is the angular momentum of the rotor and $\bar{n}$ is the average angular momentum, taken to be 0 for now. This Hamiltonian has been used to model superconducting islands connected by Josephson junctions and is also relevant to the Bose-Hubbard model. Then following [2, 5], one can examine how the incremental evolution in imaginary time

$$\langle \theta(\tau_{m+1}) | e^{-H_q \delta\tau} | \theta(\tau_m) \rangle \tag{9}$$

reduces to a classical action in two dimensions. Keep in mind that here $|\theta(\tau_m)\rangle$ denotes an entire chain of angles at a given time. Though $n_i, \theta_i$ are conjugate operators which do not commute, for small $\delta\tau$, the exponential can be broken into two pieces. Inserting a complete set of configurations $|n\rangle \equiv |\{n_k\}\rangle$ of angular momenta along the chain,

$$\begin{aligned}
\langle \theta(\tau_{m+1}) | e^{-H_q \delta\tau} | \theta(\tau_m) \rangle &= \sum_n \langle \theta(\tau_{m+1}) | \exp[-U \sum_i n_i^2 \delta\tau] | n \rangle \\
&\quad \langle n | \exp[t \sum_{<ij>} \cos(\theta_i - \theta_j)\delta\tau] | \theta(\tau_m) \rangle \\
&= \sum_n \exp[-U \sum_i n_i^2 \delta\tau] \exp[i \sum_i n_i (\theta_i(\tau_{m+1}) - \theta_i(\tau_m))] \\
&\quad \exp[t \sum_{<ij>} \cos(\theta_i(\tau_m) - \theta_j(\tau_m))\delta\tau]
\end{aligned}$$

using $\langle \theta_k | n_k \rangle = e^{in_k\theta_k}$. The Poisson summation formula

$$\sum_n e^{-Cn^2} e^{in\theta} = \sqrt{\frac{\pi}{2C}} \sum_p e^{-\frac{1}{4C}(\theta + 2\pi p)^2} \tag{10}$$

and the Villain approximation of the righthand sum as $const \times \exp[\frac{1}{2C}\cos\theta]$ yield

$$\begin{aligned}
\langle \theta(\tau_{m+1}) | e^{-H_q \delta\tau} | \theta(\tau_m) \rangle &\propto \exp[\frac{1}{2U\delta\tau} \sum_i \cos(\theta_{i,m} - \theta_{i,m+1})] \\
&\quad \times \exp[t \sum_{<ij>} \cos(\theta_{i,m} - \theta_{j,m})\delta\tau],
\end{aligned}$$

which is the transfer matrix of the (anisotropic) 2d classical XY model with periodic boundary conditions in one direction (because $\theta(0) \equiv \theta(\beta)$ in the quantum partition function).

Note that if $\bar{n}$ is allowed to be an arbitrary integer, then applying the Poisson formula with $(n-\bar{n})^2$ instead of $n^2$ merely involves shifting the dummy variable by $\bar{n}$ and produces an extra phase $e^{i\bar{n}\theta}$. Therefore, this contributes a Berry phase of $i\bar{n}(\theta(\tau_{m+1}) - \theta(\tau_m))$ to the imaginary time action. However, because of the periodic boundary conditions, the sum of all these phases is a multiple of $2\pi$ and thus does not contribute to the partition function. If $\bar{n}$ is not an integer, the classical dual of the quantum model is not evident.

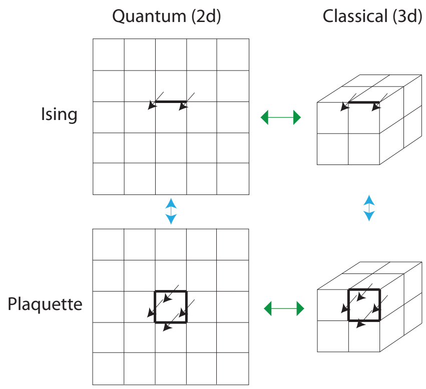


FIG. 1: Dualities both within and between quantum and classical Ising/$Z_2$ gauge models. Green arrows indicate the quantum to classical mapping; blue arrows indicate the Ising/$Z_2$ duality. In the main text, the lower right model is derived from knowledge of the other three models.

## APPLICATION TO CLASSICAL $3d$ ISING MODEL

The quantum to classical mapping becomes more involved when both sides involve gauge symmetry. I motivate this section with the following hypothetical scenario: consider the era before a classical dual of the 3d Ising model was found. However, assume that people knew not only the quantum to classical mapping but also the fact that the 2d quantum transverse Ising model

$$H_q = -J \sum_{<ij>} S_i^z S_j^z - K \sum_i S_i^x \tag{11}$$

is dual [6] to the 2d quantum gauge/plaquette model

$$\tilde{H}_q = -K \sum_{pl} \prod_{ij \in pl} \sigma_{ij}^z - J \sum_{<ij>} \sigma_{ij}^x \tag{12}$$

with the important $Z_2$ gauge constraint for every site $i$:

$$\prod_{<ji>} \sigma_{ij}^x = 1. \tag{13}$$

Here the spins lie on the bonds of a square lattice, and $pl$ denotes the smallest square of the lattice. This duality between quantum models is rather nontrivial and interesting in its own right, but for the sake of this paper I will merely assume it.

Using similar techniques as those in section 2, one can map the $2d$ quantum transverse Ising model to the $3d$ classical Ising model. Hence, applying the quantum-classical mapping to the $2d$ quantum plaquette model should yield the desired classical dual of the $3d$ Ising model. (See Figure 1 for clarity).

I now proceed to map the quantum plaquette model (12) to a classical model, following [7]. Once again, consider the incremental evolution in imaginary time

$$\langle \sigma^z(\tau_{m+1}) | e^{-H_q \delta\tau} | \sigma^z(\tau_m) \rangle \tag{14}$$

$$= \sum_{\sigma^x} \langle \sigma^z(\tau_{m+1}) | \sigma^x \rangle \langle \sigma^x | \sigma^z(\tau_m) \rangle \exp\big[\sum_l J \sigma_l^x \delta\tau\big]$$
$$\times \exp\big[K \sum_{pl} \prod_{pl} \sigma_{ij}^z \delta\tau\big] \prod_i \delta\Big( \prod_{<ji>} \sigma_{ji}^x - 1 \Big) \tag{15}$$

where I have used the shorthand $\sigma^z$ denote the collection of spins on the links which are denoted by $l$. Note also the insertion of a complete set of $\sigma^x$ states and the delta functions enforcing the gauge constraint. Each delta function can be expanded in terms of a Lagrange multiplier $\lambda_i$ that 'lives' on the link joining $(i, \tau_m)$ to $(i, \tau_{m+1})$:

$$\delta\Big( \prod_{<ji>} \sigma_{ji}^x - 1 \Big) = \frac{1}{2} \sum_{\lambda_i = \pm 1} \exp\big[i\pi \Big(\frac{1-\lambda_i}{2}\Big) \sum_{<ji>} \frac{1-\sigma_{ij}^x}{2}\big].$$

Consider a link $< ik > \equiv l$. Then

$$\langle \sigma_l^z(\tau_{m+1}) | \sigma_l^x \rangle \langle \sigma_l^x | \sigma_l^z(\tau_m) \rangle$$
$$= \frac{1}{2} \exp\big[i\pi \Big(\frac{1-\sigma_l^x}{2}\Big) \Big( \frac{1-\sigma_l^z(\tau_{m+1})}{2} + \frac{1-\sigma_l^z(\tau_m)}{2} \Big)\big].$$

Using these two expressions and summing (15) over $\sigma_l^x$ for a fixed $l$ yields

$$e^{J\delta\tau} + e^{-J\delta\tau} \sigma_l^z(\tau_{m+1}) \sigma_l^z(\tau_m) \lambda_i \lambda_k \tag{16}$$
$$\propto \exp\big[K_\tau \sigma_l^z(\tau_{m+1}) \sigma_l^z(\tau_m) \lambda_i \lambda_k\big] \tag{17}$$

where

$$K_\tau \equiv -\frac{1}{2} \ln \tanh{(J\delta\tau)}. \tag{18}$$

Note the emergence of a four-spin operator acting on a plaquette extending in the imaginary time direction. Altogether, after relabeling $\lambda$ as just another $\sigma^z$ degree of freedom,

$$\langle \sigma^z(\tau_{m+1}) | e^{-H_q \delta\tau} | \sigma^z(\tau_m) \rangle \tag{19}$$
$$\propto \sum_{\lambda, \sigma_z} \sum_{pl} K_{pl} \prod_{ij \in pl} \sigma_{ij}^z \tag{20}$$

where $K_{pl} = K_\tau$ if the plaquette extends in the imaginary time direction and $K_{pl} = K$ if otherwise.

The resulting classical model is precisely the (in this case anisotropic) $Z_2$ lattice gauge theory [4] that has been known to be dual to the $3d$ classical Ising model. Indeed, note that $J\delta\tau$ is the coupling constant of the latter theory derived from the $2d$ quantum Ising model. The relation (18) is the well-known relation between high-temperature and low-temperature expansions for both the $2d$ and $3d$ Ising models.

## FURTHER EXAMPLES AND LIMITATIONS

In addition to the three applications considered above, the QC mapping can be applied to many more models. For instance, the $1d$ quantum transverse Ising model

$$H_q = -J \sum_{<ij>} S_i^z S_j^z - K \sum_i S_i^x \tag{21}$$

can be mapped to the $2d$ classical (anisotropic) Ising model. The latter is self-dual [4], so it is expected from the mapping that the quantum model should also be self-dual. Indeed, one can perform the (non-local) transformation [6]

$$S_i^z = \prod_{j\leq i} \sigma_j^x \quad (22)$$

$$S_i^x = \sigma_i^z \sigma_{i+1}^z, \quad (23)$$

upon which the new Hamiltonian is

$$\tilde{H}_q = -K \sum_{<ij>} \sigma_i^z \sigma_j^z - J \sum_i \sigma_i^x \quad (24)$$

As the classical duality relates high temperature expansion to the low temperature expansion, the quantum self-duality similarly relates the ordered and disordered phases. Large $J$ corresponds to ordered $S^z$ and disordered $\sigma_z$, and vice versa for large $K$; thus, the quantum phase transition occurs at the self-dual point $J = K$.

It may appear that the QC mapping can be performed quite generally, yet the classical side is often complicated by the presence of a Berry phase term which has thus far no classical interpretation. Such a term already appeared in the $1d$ quantum XY model. More importantly, as I will now briefly discuss, it limits the QC mapping of many notable spin systems such as the $1d$ antiferromagnetic Heisenberg chain:

$$H_q = J \sum_{<ij>} S_i \cdot S_j. \quad (25)$$

Is there a classical model in two dimensions with the same partition function? One may be able to use the relation (1) to write down a transfer matrix in a basis of direct products of spins; however, the classical model in this case does not appear to be transparent or any more tractable than the quantum model.

A better route is to use spin coherent states to write a path integral for the spins in imaginary time. I briefly outline the results to give the reader a taste of the Berry phase. To reduce the spin Hamiltonian operator to numbers for the path integral, one can choose an overcomplete basis $|\mathbf{N}\rangle$ where each $\mathbf{N}$ is a point on $S^2$. The key feature of this basis, known as the coherent state basis, is that

$$\langle \mathbf{N}|\hat{\mathbf{S}}|\mathbf{N}\rangle = \mathbf{SN} \quad (26)$$

where $S$ is the spin of the operator representation. The above property allows one to write the incremental evolution in imaginary time as [3]:

$$\langle \mathbf{N}(\tau)| \exp\left[-\delta\tau \mathbf{H}(\hat{\mathbf{S}})\right] |\mathbf{N}(\tau+\delta\tau)\rangle \quad (27)$$

$$\approx \exp\left[-\delta\tau \langle \mathbf{N}(\tau)| \frac{d}{d\tau} |\mathbf{N}(\tau)\rangle - \delta\tau H(S\mathbf{N})\right] \quad (28)$$

Ignoring the first term in the exponential, the classical dual model is clear; one would simply have $O(3)$ rotors coupled along the spatial directions in a manner dictated by the Hamiltonian. However, the first term which is a Berry phase complicates matters; it is not only imaginary but also defined only modulo $2\pi$ for half-integer $S$ and modulo $4\pi$ for integer $S$. Such an imaginary weight is difficult to interpret for a classical partition function, making the QC mapping of little use in this case.

## DISCUSSION

Despite the obstacles presented by the Berry phase term, the QC mapping has been quite useful in mapping many standard classical models to their quantum counterparts. In this paper, I explicitly demonstrated the mapping between $0d$ quantum and $1d$ classical Ising, $1d$ quantum and $2d$ classical XY, and $2d$ quantum and $3d$ classical Ising models. The essence of the QC mapping has also been applied in Monte Carlo simulations for quantum systems.

Practicality aside, the QC mapping is quite remarkable, relating quantum and classical theories in different dimensions. At the core of the mapping is the path integral perspective of the quantum partition function and the transfer matrix perspective of the classical partition function. Given that the energy scale and temperature of the quantum system are manifested in the inverse correlation length and size of the classical system, one may wonder if other properties of the quantum system may find interpretation in the classical side. For instance, I wonder if the QC mapping can provide classical intuition on entanglement entropy [8], a measure of the entanglement between two quantum subsystems. Such questions and interpretations of the Berry phase term are left for the future.

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