A Chern-Simons treatment of abelian topological defects in 2D

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The non-existence of long range order in $d \leq 2$ has lead to a lot of topological studies in these dimensions in the past two decades. Here we are going to apply a well-known method in 2D called Chern-Simons field theory, which play a crucial role in fractional quantum Hall effect and conformal field theory, to the case of topological defects to obtain a clean gauge theory description for dealing with these objects.

Keywords: topolgical defects, Chern-Simons theory, Kibble mechanism, gauge theory

INTRODUCTION

Here we are going to give a brief review of the general theory of topological defects and give a Chern-Simons field theory prescription of topological defects in special case of 2D. In the first part we will discuss some topological concepts related to fundamental group to be able to categorize different order parameter spaces. We will prove that order parameters can in general be discontinuous leading to phenomenon of topological defects. We will also give a list of possible topological defect in fully translational symmetric systems.
In second part we will discuss in detail the creation of string defects in superfluid ^{4}He as a prototype to all second order phase transitions which give rise to monopole defects in 2D. Then using an analogy between the physics of the defects of superfluid ^{4}He and $U(1)$ gauge theory we will construct a general effective gauge theory, called Chern-Simons field theory, which describes these topological defects in 2D.

DISCONTINUOUS ORDER PARAMETER

Fundamental Group

In general the main concern of topology is whether two different objects can be deformed continuously to each other or not. In other words topology looks for ways to categorize all objects in the world based on their formation properties. There are at least two different notions of continuity in topology: one is homeomorphism which is a function from A to B, $f : A \to B$, such that f is a continuous bijective function with the additional property that f^{-1} is also bijective. Intuitively speaking two objects are homeomorphic to each other if we can deform them into each other without ripping them apart or gluing one part to the other; but stretching, twisting and shrinking are allowed. Basically we say two objects are topologically the same if they are homeomorphic. Based on this we can define topological space, if we can consistently define the open sets of the space such that the continuity is well-defined for a function from one topological space to another.
The other continuity notion is homotopy. Fixing a topological space we say two "loops" with a point in common, say γ_1 and γ_2 are homotopic if we can continuously change $\gamma_1 \to \gamma_2$, but not necessarily the other way around. Hence although we are still not allowed to rip a loop, we can glue, e.g. in $\mathbb{R}^2$ a circle and a point are homotopic, but not homeomorphic.
Denoting the set of all loops passing through some point x by Γ_x, the homotopy divides this set into equivalence classes called homotopy classes, such that in each homotopy class all elements are homotopic to each other. As a result in a topological space Ω one can make a new set, the fundamental set, $\pi_1(x, \Omega)$ containing these homotopy classes. Defining the product of two loops with common point x by extending the first loop by the second one we can promote this fundamental set into the *fundamental group* [1].
Since topology is a the mathematics of mapping objects to another, the concept of invariance is of special importance here. A topological invariant is a property which is preserved under hemeomorphism. A theorem in topology states that if the topological space Ω is connected then the fundamental groups of different points x, $\pi_1(x, \Omega)$ are isomorphic to each other and basically the fundamental group is independent of x. Another important theorem in topology [2] states that in connected spaces where notion of fundamental group is globally the same, $\pi_1(\Omega)$ is a topological invariant. An immediate consequence of this fact, which is relevant to our study of order parameters, is that if two space have different fundamental groups then there is no bejctive continuous function from at least one space to the other. Since physical topological spaces are almost always connected these theorems are significant to us.

Order Parameter

In Landau-Ginzburg theory phase transitions are almost always accompanied with a symmetry breaking. As usually envisaged, the breaking of a symmetry implies a sudden change from a symmetric state to an ordered state, in which some symmetry-breaking order parameter acquires a non-zero value. In other words, a phase transition is a process in which the order parameter balances the external influences to the physical system with the internal tendency for the lowest energy configuration. For continuous, or second order, phase transitions one has in mind a smoothly changing free energy density with an associated order parameter that transforms from a symmetric to a non-symmetric configuration.

The order parameters are in general fields (scalar, spinor, vector, etc.) over the lattice or continuum. The question then becomes is this order parameter unique? Each order parameter gives rise to a Landau-Ginzburg free energy, so although order parameter is far from unique, we choose the order parameter corresponding to the least energy configuration as the physical one. We also postulate that order parameters giving the same energy configurations are equivalent. As a result an order parameter, φ, can be considered also as a map from M, which is the physical continuum or lattice, to S, the space of minima. [3]

Topological Defects

A way of thinking of $\varphi : M \to S$ is to attach a space S (fiber) over each point of M, making a bundle. Then in mathematical terms the order parameter field is a section of this fiber-bundle, meaning on each point of M, say $x \in M$, φ takes a value in the fiber S directly above x. Then we can define the continuity of section φ in the sense of fiber-bundles, which is roughly saying that if we choose y is in a infinitesimal neighborhood of x in M, then φ is continuous if by pulling back $\varphi(x)$ from S_x to a S_y, the pulled back $\varphi(x)$ is lies near $\varphi(y)$. Now by taking a loop ℓ in M, we are corresponding a loop L in S given by the restricted section $\varphi|_\ell : \ell \to S$. But then the question become "is the fundamental group $\pi(M)$ and $\pi_1(S)$ the same"? The answer is no, not necessarily, since there is no apparent connection between M and S. An immediate example is the X-Y model in which S is a circle S^1, while M (forgetting about the lattice for now) is $\mathbb{R}^2$. Every loop in $\mathbb{R}^2$ can be shrunk to a point so $\pi_1(\mathbb{R}^2) = 1$ while we can have a loop going around the circle n-times without us being able to shrunk it to a point, so $\pi_1(S^1) = \mathbb{Z} \neq \pi_1(\mathbb{R}^2)$. Utilizing the theorem we mentioned in fundamental group section this means that order parameter is discontinuous at least in a submanifold $C \subset M$. This C being empty or not depends on the topology order parameter space and distribution of order parameter in the configuration.

Restricting φ to $M - C$ we have a continuous order parameter, so C must be some kind of defect! Given the nature of the derivation of this defect we call this a topological defect. In general if we take $M = \mathbb{R}^d$ or any other simply connected manifold, as usually is the case in physics then the necessary condition for existence of topological defects becomes

$$\exists k : \pi_k(S) \neq 1 \tag{1}$$

where π_k is just the generalization of fundamental group from loops (homotopic to S^1) to k-dimensional objects homotopic to S^k, and is called kth homotopy group. If this condition is present topological defect can happen but this condition is by no means a sufficient one. We will talk about formation of defects (the sufficient condition) later in the paper, but for now suffice to say that defects usually happen below some critical temperature due to non-equilibrium processes.

Types of Topological Defects

In this article we are going to discuss systems in which even after the phase transition the full translational symmetry is preserved. In other words we won't worry about the crystalline structure. Treating these kinds of topological defects are a little bit tricky. We basically have to forget about the rotational symmetry and replace it by an orientational symmetry plus translational one. Then the order parameter becomes the dislocation and considering the equivalence between lattice points the order parameter space becomes a d-dimensional torus in the easiest possible case. Since translational symmetry will break semi-infinite defects which are not closed are possible in this case (a line defect). The interested reader is referred to Mermin's review paper [4].

The possible candidates for translational symmetric defects in 3D then are the following

- *Point defects* or *monopoles*
- *String defects*
- *Domain walls* or *kinks*
- *Texture* or *Skyrmion*

Strings (domain walls) are either closed loops (surfaces), or they end at the boundaries of the condensed matter system. In 3D monopoles have $\pi_1 = 1$ and $\pi_2 \neq 1$, while in 2D $\pi_1 \neq 1$ and other homotopy groups are undefined. Domain walls always have an Ising order parameter as we will show later on, and string defects in d-dimensions are related to monopoles defects in $d - 1$ dimensions.

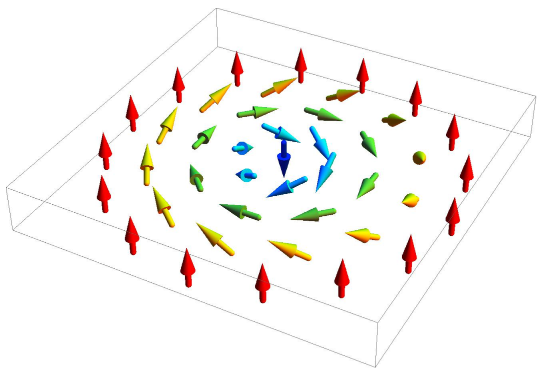

FIG. 1: Going through the disk the order parameter sweeps the whole S^2 once.

Skyrmions To explain what is really happening here let's go to 2D for convenience. Here there are only three types of topological defects: monopole, domain walls and

Skyrmions. The string case is absent since string defects and domain walls are basically the same in 2D. The case we haven't yet discussed is $\pi_2 \neq 1$ in 2D while $\pi_1 = 1$. An immediate candidate for such a system is 3D unit spin order parameter on 2D lattice/continuum. Since this order parameter is on a S^2 naturally we cannot shrink any 2D sphere-like object to a point on it and it will give rise to $\pi_2(S^2) \neq 1$, but any loop on sphere can be shrunk to a point so $\pi_1(S^2) = 1$. In order to have such a defect for at least one disk (not a circle!) on the lattice the order parameter should cover whole of S. Such a distribution of order parameter is shown in Fig. 1.

DEFECT FORMATION IN 2-DIMENSIONS: KIBBLE MECHANISM & CHERN-SIMONS FIELD THEORY

During any phase transition at finite temperature, defects will be produced due to thermal fluctuations. Apart from this thermal production, there is a *non-equilibrium process* which dominates at low temperatures. This process of defect formation is generally known as the Kibble mechanism and arises due to a sort of domain formation after the phase transition with defects forming at the junctions of these domains [5, 6]. We will first discuss the string defects of superfluid ^{4}He [7] to introduce the Kibble mechanism. Then will try to generalize this with a field theoretic tool called Chern-Simons effective Lagrangian.

String Defects of Superfluid ^{4}He

The Landau-Ginzburg free energy of superfluid ^{4}He is

$$F = K \left|\nabla\psi\right|^2 - \alpha \left|\psi\right|^2 + \beta \left|\psi\right|^4 \qquad (2)$$

where the order parameter, ψ is a quantum mechanical complex wave-function, $\psi = re^{i\phi}$. In the normal phase $(T > T_c)$ $\alpha < 0$ and $\langle\psi\rangle = 0$. On $\alpha = 0$ $(T = T_c)$ a second order phase transition happens making $\langle\psi\rangle = ne^{i\phi}$ where $n = \sqrt{\beta/\alpha}$. The effective order parameter in this phase is the quantum mechanical phase, ϕ. Kibble postulated that if we choose two points with distances much higher than the correlation length, ξ, the phases at those points are also uncorrelated. Consequently if we take a very large circle in the superfluid, ℓ_{big} and divided the circumference into domains of length $\sim \xi$, then on each domain the phase is chosen randomly and independent of the other domains. Now if the circle is large enough there is enough randomization which we can safely assume the the order parameter space which is S^1 is completely covered gooing around the loop. So for ℓ_{big} we obtain a non-unity element L_{big} of $\pi_1(S^1)$ or in technical terms a loop with non-zero winding number.
The winding number is an integer determining how many times we loop around the multiply connected order parameter space. As a result if we change ℓ_{big} in any continuous way, the winding number cannot change. This due to the fact that integers don't vary continuously, so they should not vary at all in continuous deformation. In other words the winding number of the order parameter loop is a topological invariant. Now if shrink ℓ_{big} to a point since the gradient term in free energy should remain finite we must have $\langle\psi\rangle = 0$ on this point. So below T_c there necessarily exist a string defect which is in normal phase and the order parameter winds around this string.
More generally Kibble mechanism states that around the critical temperature in the disordered phase, tiny bubbles of ordered phase form, randomly in space and independently of each other. As the system is more and more cooled down the number of these bubbles increase up to the point T_c which they become dense and merge together making the ordered phase. The advantage of this mechanism is the guarantee of randomness of the order parameter for points far from each other.

An Analogy to $U(1)$ Gauge Theory

The phenomenon in superfluid ^{4}He is reminiscent of Aharonov-Bohm effect in which two electrons curling around a confined and local magnetic flux (a solenoid) in opposite directions destructively interfere upon meeting again. In other words the phase of a fermion changes by 2π when winding around a magnetic vortex. Taking $c = \hbar = 1$, in general

$$\psi(\text{final}) = \exp\left[-ie \oint \vec{A}\cdot d\vec{\ell}\right]\psi(\text{initial}) \qquad (3)$$

This also tells us that magnetic flux is quantized and $\Phi = 2\pi n/e$, which is exactly what we mean by a vortex. This effect is understood via $U(1)$ gauge theory of QED. Due to EM gauge freedom on each point of space-time x, we should put a fiber consisting of all of the possible values of $A(x)$. Then the gauge fixing condition defines a section of this fiber-bundle which describes the physical system. This section assigns a vector field on space-time manifold. The covariant derivative respecting gauge symmetry then becomes

$$D_\mu\psi = (\partial_\mu - ieA_\mu)\psi$$

The fiber-bundle language of gauge theory is a geometrical one, so it is no surprise that if we call A the connection and $F_{\mu\nu}$ the curvature tensor [8]. I'm not going into the details of gauge theory here since all we need is $U(1)$ gauge theory and many of the complications of gauge theory can be avoided in this case. I just wanted to introduce the terminology and show how the Aharonov-Bohm phase arises from the covariant derivative.

Chern-Simons Theory in 2-Dimensions

Suppose we take a Lorentz (or Galilean) invariant Lagrangian for some 2D gauge field a_μ

$$\mathcal{L} = \frac{\kappa}{2}\epsilon^{\mu\nu\lambda}a_\mu\partial_\nu a_\lambda \tag{4}$$

This is very different from general $-F_{\mu\nu}F^{\mu\nu}/4$ Lagrangian and is specific to 2-dimensions (We can be generalized to any $2k{+}1$ dimensions, but then this it becomes kth order in derivatives). Also under a gauge transformation $a_\mu \to a_\mu + \partial_\mu\Lambda$

$$\delta\mathcal{L} = \frac{\kappa}{2}\partial_\mu(\Lambda\epsilon^{\mu\nu\lambda}\partial_\nu a_\lambda) \tag{5}$$

which shows that under usual boundary conditions this theory is gauge-invariant too. This is called a (abelian) Chern-Simons Lagrangian. We also changed notation from A to a to emphasize the fact that a is not necessarily an EM gauge field but it is $U(1)$ (hence the abelian adjective). Coupling this term to a source

$$\mathcal{L} = \frac{\kappa}{2}\epsilon^{\mu\nu\lambda}a_\mu\partial_\nu a_\lambda - a_\mu J^\mu \tag{6}$$

yields a curvature tensor $f_{\mu\nu} = \frac{1}{\kappa}\epsilon_{\mu\nu\lambda}J^\lambda$, or equivalently a Chern-Simons effective magnetic field

$$B_{\text{CS}} = \frac{\rho}{\kappa} \tag{7}$$

which is perpendicular to the 2D surface. Using the gauge invariance we can go to Coulomb gauge, $\nabla\cdot\vec{a} = 0$, and use the usual Biot-Savart law since $B_{\text{CS}} = \nabla\times\vec{a}$. Also writing the source term as $\rho(\mathbf{r}) = \sum\delta(\mathbf{r}-\mathbf{r}_i)$, we obtain

$$\vec{a}(\mathbf{r}) = \frac{1}{2\pi\kappa}\sum_i \frac{\hat{\mathbf{z}}\times(\mathbf{r}-\mathbf{r}_i)}{|\mathbf{r}-\mathbf{r}_i|^2} = \frac{1}{2\pi\kappa}\nabla\sum_i \arg(\mathbf{r}-\mathbf{r}_i) \tag{8}$$

Assuming there is only one monopole (which is equivalent to having a string defect in 3D as we will see later) this means $\vec{a}$ is necessarily winding around that monopole. Also since a_0 is not as always dynamic if we assume for simplicity that $J_i = 0$ then $a_0 = 0$.

Back to Discontinuous Order Parameter

In the relation we found for $\vec{a}$ we notice that $\arg(\mathbf{r}-\mathbf{r}_i)$ has the same kind of discontinuity the order parameter had in case of the string defects for superfluid ^{4}He. But string defects in 3D are effectively two dimensional since one can ignore the dimension along the string. Then the theory becomes the same as monopole defects in 2D. This observation suggests that we can construct a effective Chern-Simons field theory describing monopole defects by (φ is the order parameter)

$$\vec{a} = \frac{1}{2\pi\kappa}\nabla\varphi \tag{9}$$

Under this prescription one can go back and forth between order parameter and the gauge field. The agent responsible for creation of monopole defects are penetrating confined and local Chern-Simons effective magnetic fields or vortices, just like the case of EM (Aharonov-Bohm effect). Also one can identify the fiber-bundle of order parameter with the gauge fiber-bundle, unifying the two pictures.

Chern-Simons terms have no effect until a localized vortex appears in the system. In other words when we are cooling the system the randomly created bubble begin to enlarge by coalescing with each other. In this process sometimes a hole of disordered phase gets surrounded by a large ordered bubble. Ordinarily on the point of criticality one expects these holes to dissolve but due to topological considerations which we discussed above some of these hole cannot do that without ruining the whole phase and stay in their original normal phase. Since annihilation of these defects will result in a global catastrophe, this is not something which can be corrected in the system by local deformation. The defects can move but they cannot be destroyed.

Now considering a domain wall since we can make a wall by putting a lot of these monopoles near each other we see by arguments simialr to electromagnetism that the order parameter or the magnetic field of this system can only take two values $\pm\sigma$. This is an Ising model so domain walls in general have an Ising order parameter. The consideration of domain wall which begin and end on the boundary of the system are pretty much the same in any dimensions and effectively two dimensional. The closed domain walls on the other hand are another story.

CONCLUSIONS

The advantage of Chern-Simon field theory is that utilizing the gauge theory analyzing the system becomes much simpler. For example seeing that monopoles interact with each other by a Coulomb potential is immediate since this is just an analog of electromagnetism. Connections to quantum Hall effect are also quite clear in this picture by the same line of reasoning (magnetic vortices are basically the same as Chern-Simons vortices).

The huge downside of this theory is that it is limited to 2D only. It is not possible to write a clean gauge theory like Chern-Simons in other dimensions. This is related to Conformal Field theory and the fact that 2D is basically the same as complex numbers. But scaling is not a well-defined symmetry in dimensions other than 2. In such dimension we should rely on Renormalization groups and other field theory methods.

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