

The Ising Model on a Binary Tree

Jonathan Mendoza*

MIT Department of Mechanical Engineering

(Dated: May 17, 2014)

We present the results of a numerical calculation of the Ising Model on a binary tree. Low energy excitations are identified as a region of flipped spins of a node and all of its descendants in the hierarchy. The low energy excitations keep the net magnetization from approaching unity at temperatures below the critical value. The critical temperature (T/J) is also found for a tree depth of ten to be 1.5 which is lower than the analytical value for an infinite depth tree. This discrepancy stems from the fluctuations of the nodes at the bottom of the finite depth tree.

1. INTRODUCTION

The Ising Model is one of the earliest conceived models that exhibits a phase transition. As a toy model to describe magnetism, the Hamiltonian for the Ising Model can be expressed as

$$-\beta H = J \sum_{\langle i,j \rangle} \sigma_i \sigma_j \quad (1)$$

where the sum runs over the nearest neighbors. While the solution to the 2D square lattice can be solved by the transfer matrix method which was pioneered by Onsager, exact solutions exist on other lattices as well. In this paper we will discuss the numerical results of the Ising model on a binary tree.

The binary tree is a specific form of a Bethe lattice with a connectivity of two. From a computer science perspective one can define an object known as a node that has a parent node and two child nodes. The Ising model is implemented on this tree structure by placing bonds of equal strength between a node and its parent and children. If one were to allow for bonds between the two children of a given node, this lattice would simply reduce to a finite sized triangular lattice which would break some of the interesting low temperature phenomena which will be discussed.

In order to sample the Ising model at a finite temperature, the metropolis algorithm is used [1]. The procedure begins with a random spin configuration to represent a high temperature state. A point on the lattice is then chosen at random. If the energy of the configuration reduces when the randomly chosen spin is flipped, then accept the change. If the energy increases, accept the flip with probability $e^{-\beta \Delta E}$. By computing the net magnetization over flip iterations (the sum over the Ising variables), we can keep track of the relaxation to a thermal state and identify critical behavior.

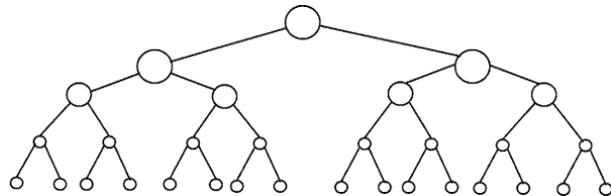


FIG. 1: Visual representation of a binary tree. The root of the tree is the topmost node while the leaves are defined as the bottom most nodes.

2. CALCULATIONS

This section will begin with the discussion of the low temperature limit. When below the critical temperature, the net magnetization will thermalize to a non-zero value. Unlike the 2D-ising model, for non-zero temperatures, the net magnetization is not necessarily of order unity as depicted in Fig. 1. This low-temperature behavior stems from the connectivity of the lattice.

If we consider the ground state where all of the spins are aligned then the lowest energy excitation is a flipped spin at the bottom of the tree since the only bond that exists is with its parent node. In the limit of an infinite binary tree where we cannot simply flip the lowest spin on the tree, the lowest energy excitation is a configuration where a node and all of its children and their children are flipped as well. Such states can lead to magnetization far from unity. The most extreme example of such a configuration stems from a flipped spin of a child of the root of the tree and all of its children. This results in a configuration energy only two units larger than the ground state but with a magnetization of $1/N$.

Lets compare this to the low energy excitations of the 2D square lattice where low temperature excitations are droplets of opposite spin where the energy scales with the perimeter of the droplet [2]. The partition function can be expressed as

$$Z = e^{2NK} \sum_{\text{Islands of } (-) \text{ spins}} e^{-2K \times \text{perimeter}} \quad (2)$$

Unlike the excitations in the binary tree where the lowest energy excitations can be of arbitrary size, the cost

*Electronic address: mrfox321@mit.edu; URL: <http://web.mit.edu>

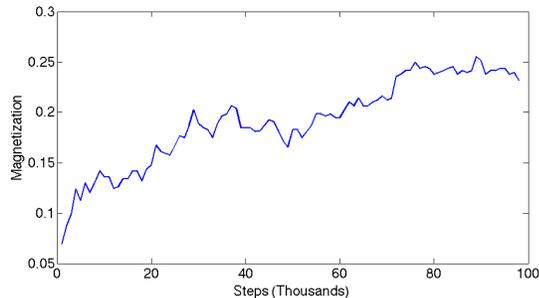


FIG. 2: Magnetization of the binary tree as a function of spin flip operations. The temperature (T/J) for this simulation is 0.5. Low energy excitations prevent the magnetization from asymptoting to values on the order of unity.

scales with the size of the droplet for the 2D square lattice. Size dependence of the excitations can be included by introducing the field term

$$-\beta H_{field} = h \sum_i \sigma_i \quad (3)$$

which selects for smaller regions of opposing spin.

The evolution of the net magnetization as the system is quenched is shown in Fig. 3. For a tree depth of 10, the critical ratio of temperature to critical temperature (T/J) is found to be about 1.5. The infinite Bethe lattice has a zero-field critical value of 1.85 [3]. The discrepancy comes from the fluctuations of the bottom nodes since they only have one bond, allowing for low cost fluctuations. At temperatures just below the critical point, the fluctuations stem from the low energy excitations that carry large magnetization. Although not shown, other calculations capture the net magnetization drop to nearly zero despite being below the critical temperature. This stems from the aforementioned scenario where the left and right half of the tree are of opposite values.

3. CONCLUSIONS

This paper discusses the results of the Ising model implemented on a binary tree, a subset of the Bethe lattice. The main findings are the discrepancies between the low energy excitations of the binary tree and a generic 2D lattice with translational symmetry. Whereas a 2D translationally symmetric lattice has droplets of opposing spin whose energy scales with the perimeter of the droplet, the low energy excitations on the binary tree are excitations where a node and all of its children and their children are flipped. This allows for arbitrarily sized excitations since the energetic cost comes from the parent node. This results in a net magnetization below the critical temperature to deviate from unity. A low energy configuration exists with a net magnetization of $1/N$ where the left half and right half of the binary tree are opposite spins. The

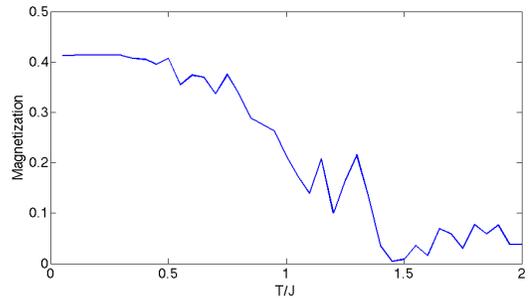


FIG. 3: Magnetization as a function of temperature. The simulation is quenched from an initial value of 2 and decreases in increments of 0.05. The magnetization does not asymptote to unity due to the low energy excitations that carry large opposing magnetization.

critical temperature is slightly lower than the exact solution of the infinite depth tree due to the lowest nodes only having one bond.

4. BIBLIOGRAPHY

- [1] Fricke, Tobin. "Monte Carlo investigation of the Ising model." (2010).
- [2] Kardar, Mehran. Statistical physics of fields. Cambridge University Press, 2007.
- [3] Bruinsma, R. "Random-field Ising model on a Bethe lattice." Physical Review B 30.1 (1984): 289.