Vortex-Magnetic Field Duality in the Classical XY Model

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Abstract

We here demonstrate a useful duality between the compact (*i.e.* supporting vortices) XY model and the non-compact XY model with a magnetic field. We calculate the partition function for each model and show that they are related by duality. This duality allows us to answer important questions about each model by looking at its dual.

1 Introduction

Dualities play a central role in physics, from the basic notion of the Fourier transform to the most abstract dualities of string theory. In the field of statistical physics, dualities are particularly important and useful. They often allow us to relate a model describing certain physical variable to a model describing different, seemingly unrelated variables. Here, we will demonstrate a duality which is particularly interesting, for the reason that it relates a model with a compact variable to a model with a non-compact variable. This duality is between the compact XY model (the one we usually study), and the non-compact XY model with a magnetic field. We look at the latter first.

2 Non-Compact XY Model with a Magnetic Field

Let us consider the action for the XY model, in the continuum limit, with a magnetic field along a specific direction, breaking the original $U(1)$ symmetry of the problem:

$$\beta H = \int d^2x \left(\frac{K}{2} (\nabla\theta)^2 - h\cos(\theta) \right) \tag{1}$$

This can also be viewed as the action for the $n = 1$ case of the $\mathbb{Z}_n$ clock model, which in general has $n\theta$ instead of θ in the argument of the cosine. Importantly, in this case we let the variable θ be non-compact, *i.e.* we do not identify $\theta = 0$ with $\theta = 2\pi$ (or any other value for that matter). Each value is distinct, so the range of θ is truly isomorphic to $\mathbb{R}$. This prevents the system from supporting any topological excitations, since the fundamental group of the real numbers is trivial, $\pi_1(\mathbb{R}) = 0$, as $\mathbb{R}$ is simply connected. Thus, we can take all θ configurations to be smoothly deformable to a constant configuration, and thus accessible via perturbation theory, so we are free to expand functions of θ in Taylor series, and the partition function will have no contribution from vortices. With this in mind, we write the partition function for the theory as:

$$Z = \int \mathcal{D}\theta e^{-\beta H} = \int \mathcal{D}\theta e^{-\frac{K}{2}\int d^2x(\nabla\theta)^2} \exp\left(\frac{h}{2} \int d^2x (e^{i\theta} + e^{-i\theta}) \right) = \tag{2}$$

$$\int \mathcal{D}\theta e^{-\frac{K}{2}\int d^2x(\nabla\theta)^2} \frac{\int \mathcal{D}\theta e^{-\frac{K}{2}\int d^2x(\nabla\theta)^2} \sum_{n=0}^{\infty} \frac{1}{n!}(\frac{h}{2}\int d^2x(e^{i\theta}+e^{-i\theta}))^n}{\int \mathcal{D}\theta e^{-\frac{K}{2}\int d^2x(\nabla\theta)^2}} = \tag{3}$$

$$Z_0 \sum_{n=0}^{\infty}\sum_{k=0}^{n} \frac{1}{k!(n-k)!}\left(\frac{h}{2}\right)^n \left(\Pi_{i=0}^{n}\int d^2x_i\right)\langle \exp(i(\theta(x_1)+\theta(x_2)+...+\theta(x_k)-\theta(x_{k+1})-...-\theta(x_n))\rangle_0 \tag{4}$$

where Z_0 is the partition function when $h=0$, and the correlations with subscript 0 are taken at $h=0$. From the roughening transition problem encountered in the course, we know that the correlation will vanish unless we have equal numbers of positive and negative signs in the argument of the exponent, *i.e.* $k=n/2$. This result follows from the invariance of the unperturbed Hamiltonian under the transformation $\theta \to \theta + a$ for constant a. Thus, we have, after performing the normal Gaussian integral in two dimensions:

$$Z = Z_0 \sum_{n=0,\text{even}}^{\infty} \frac{1}{(n/2)!(n/2)!}\left(\frac{h}{2}\right)^n \left(\Pi_{i=0}^{n}\int d^2x_i\right) \exp\left(\frac{1}{2\pi K}\sum_{i<j} q_i q_j \ln(|x_i - x_j|)\right) = \tag{5}$$

$$Z_0 \sum_{k=0}^{\infty} \frac{1}{(k!)^2}\left(\Pi_{i=0}^{2k}\int d^2x_i\right) \exp\left(2k\ln(h/2) + \frac{1}{2\pi K}\sum_{i<j} q_i q_j \ln(|x_i - x_j|)\right) \tag{6}$$

where $q_i = 1$ for $1 \le i \le k$ and -1 for $k+1 \le i \le 2k$.

3 Compact XY Model without a Magnetic Field

Now, let us examine the same model, but with h set to 0, restoring the $U(1)$ symmetry, and taking θ to be compact, *i.e.* identifying $\theta = 0$ with $\theta = 2\pi$, thus allowing the existence of vortices in the system. (We will refer to the coupling in this case as $\tilde{K}$, to distinguish the two systems.) Before accounting for vortices, the partition function is the same Z_0 found in the previous section, except with the constant $\tilde{K}$, so we will denote this as $\tilde{Z}_0$. As we have seen in the course, the smooth part and the topological part of θ decouple, allowing us to write the full partition function as the product of Z_0 and the partition function for the vortex configurations:

$$Z = \tilde{Z}_0 \sum_{n,m=0}^{\infty} \frac{1}{n!m!}\left(\Pi_{i=0}^{m+n}\int d^2x_i\right) \exp(-\beta E(x_1,...,x_n,x_{n+1},...,x_{n+m})) \tag{7}$$

where $E(x_1,...,x_n,x_{n+1},...,x_{n+m})$ represents the energy associated with a configuration of n vortices at locations $x_1,...,x_n$ and m anti-vortices at locations $x_{n+1},...,x_{n+m}$. The factors of $n!$ and $m!$ are present to account for the fact that the exchange of two vortices does not lead to a distinct state of the system, so we can think of vortices as particles obeying statistics appropriate for identical particles. As we know, if the charges are unbalanced, then the energy cost will depend logarithmically on the size of the system, which makes such configurations too energetically costly to make a significant contribution to the partition function. The energy associated with an equal number of vortices and anti-vortices has been calculated during the course, and contains both a core energy contribution $E_0 \equiv -\frac{1}{\beta}\ln y_0$ from each vortex and anti-vortex, and also a Coulomb interaction between pairs of vortices.

Changing notation to match the previous section, we have:

$$Z = \tilde{Z}_0 \sum_{k=0}^{\infty} \frac{1}{(k!)^2} \left(\Pi_{i=0}^{2k} \int d^2 x_i \right) \exp \left(2k \ln y_0 + 2\pi \tilde{K} \sum_{i<j} q_i q_j \ln(|x_i - x_j|) \right) \tag{8}$$

where $q_i = 1$ for a vortex ($1 \leq i \leq k$) and -1 for an anti-vortex ($k+1 \leq i \leq 2k$).

4 Duality Relation

As can be seen from equations 6 and 8, the partition functions for these two models have exactly the same form. Z_0 and $\tilde{Z}_0$ are analytic functions (there is no phase transition in the non-compact XY model without a magnetic field), so we can ignore this part. We then see that the important part of the partition functions will match up exactly if we identify:

$$y_0 = h/2 \qquad 2\pi K = \frac{1}{2\pi \tilde{K}} \tag{9}$$

We therefore see that the compact XY model without a magnetic field has the same phase transition behavior as the non-compact XY model with a magnetic field, so the two models can be considered as dual to each other. In particular, a compact system with a very small vortex core energy E_0, *i.e.* large y_0, is dual to a non-compact system with a very large magnetic field. When E_0 is small, vortices are energetically very cheap and can proliferate through the system and destroy the algebraic long-range order, indicating a high-temperature phase of the compact system. On the other hand, when h is very large, the magnetic field is strong enough to polarize all spins along its direction, corresponding to an ordered low-temperature phase. The relation between K and $\tilde{K}$ similarly tells us that when one is small, the other is large, so the compact model at high temperatures maps onto the noncompact model at low temperatures, and vice versa.

This duality allows us to obtain the criticality of one model from the other with basically no work. We know that the compact XY model has its critical point at $\tilde{K} = 2/\pi$, $y_0 = 0$, so the non-compact XY model is critical at $K = 1/8\pi$, $h = 0$. (Note that the difference between this and the critical value obtained in the $n = 1$ roughening transition problem is due to the

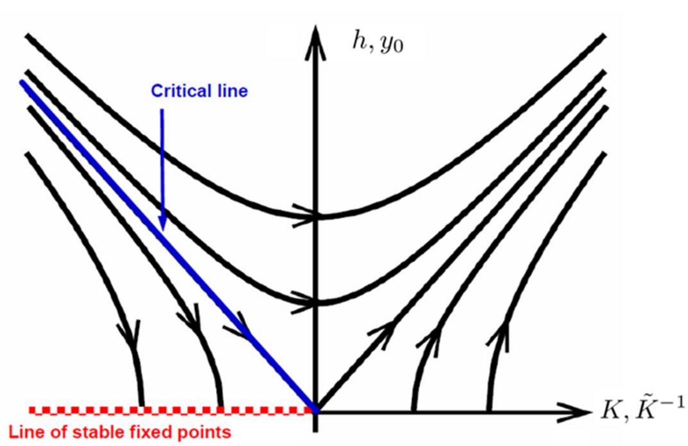

Figure 1: RG flows for the compact and non-compact models

difference in arguments of the cosine, $\cos(\theta)$ in this case and $\cos(2\pi\theta)$ in the other, reflecting a field redefinition.) We can also use the duality relations to obtain the renormalization group flow equations for one model from the other, since the duality relations hold for couplings specified with the same cutoffs. For example, in the case of the compact XY model, we have the standard RG equations of the BKT transition:

$$\frac{d\tilde{K}^{-1}}{d\ell} = 4\pi^3 a^4 y_0^2 \qquad \frac{dy_0}{d\ell} = (2 - \pi\tilde{K})y_0 \tag{10}$$

Using the duality relations, Equation 9, we immediately have:

$$\frac{dK}{d\ell} = \frac{1}{4\pi^2}\frac{d\tilde{K}^{-1}}{d\ell} = \pi a^4 y_0^2 = \frac{1}{4}\pi a^4 h^2 \tag{11}$$

$$\frac{dh}{d\ell} = \left(2 - \frac{1}{4\pi K}\right)h \tag{12}$$

which, up to redefinitions of the couplings by constant factors, are exactly the RG equations obtained in the context of the roughening transition. The only significant difference between the two flows is the exchange of K for $\tilde{K}^{-1}$, and flow in K, h space will look qualitatively the same as the flow in $\tilde{K}^{-1}, y_0$ space, as depicted in Figure 1.

Of course, if we had solved the non-compact model first, we could reverse the logic and use that solution to obtain the critical point and RG flow equations of the compact XY model. In a sense, this is the more natural path of logic. In general, a non-compact model is easier to solve directly than its compact counterpart, precisely because there are no topological excitations. Thus, the analysis of the non-compact model, as we undertook in the case of the roughening transition, provides a fairly efficient way to obtain the physics of the BKT transition, without going through all the trouble of dealing with vortices.

5 Conclusion

We have here demonstrated, using a path integral partition function approach, that a duality exists between the compact XY model and the non-compact XY model with a magnetic field. In passing, we also note that such a duality can be constructed directly from the compact XY model on a lattice by introducing non-compact variables on an appropriately chosen dual lattice. In general, when faced with a compact model, it is often a fruitful strategy to look for some such duality transformation to a non-compact model. In this case, the duality gives us nearly all of the important physics of the compact model. The non-compact results tell us the critical point and RG flow equations of the compact model, which allow us to extract all critical behavior, such as critical exponents and correlation length. We therefore see that duality is an extremely powerful tool for handling the XY model.

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