Summary of several analytical studies of the BEG model

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The Blume-Emery-Griffiths (BEG) model is a Ising model with spin-1 particles and nearest neighbour biquadratic interactions. It was initially proposed by the three authors as a generalization of the Ising model to study the λ phase transition of He^4 and phase separation in $He^3 - He^4$ mixtures. Later many authors continued to study mutated versions of the BEG model, such as in 3-d, with repulsive biquadratic interaction, etc. Also people studied the BEG model on different lattices, using typical methods of exact solutions, position space renormalization group, and Monte Carlo simulations. In this paper I will try to summarize partly the analytical studies on BEG model with several selected papers. These papers present results for different lattices, and can give us a rough and broad view of the BEG model.

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THE INITIAL BEG MODEL

In the initial paper by Blume, Emery & Griffiths [2], they were motivated by the Ising model doped with a certain concentration of nonmagnetic impurities as a simplified model to simulate the superfluidity and phase separation in a $He^3 - He^4$ mixture. This analogy comes from the fact that both ferromagnetic transition and the superfluid transition are typical second order phase transitions and presents similar universalities, and He^3 component doesn't contribute to the superfluidity, as the impurities in ferromagnetic transition doesn't contribute to the magnetization. The He^4 component is represented by spins of ± 1, while the He^3 component corresponds to the non-magnetic impurities with spin 0. The total spin

$$M = \frac{1}{N}\sum_{i=1}^{N} S_i$$

corresponds to the original superfluid order parameter in liquid helium. The He^3 concentration

$$x = \frac{1}{N}\sum_{i=1}^{N}\left(1 - S_i^2\right)$$

now is another order parameter which will determine whether the superfluid phase separation can happen. With this analogy between ferromagnetic and superfluidity, the Hamiltonian for nearest-neighbour interactions is written as:

$$H_S = -J\sum_{<i,j>} S_i S_j \tag{1}$$

which from known results of Ising model will lead to a second order phase transition at a certain critical temperature T_c if no impurities included. T_c will intuitively decrease as the impurity concentration increases. To model the phase separation, they proposed an interaction term in the Hamiltonian:

$$\begin{aligned} H_I = &- K_{33}\sum_{<i,j>}(1-S_i^2)(1-S_j^2) - K_{44}\sum_{<i,j>} S_i^2 S_j^2 \\ &- K_{34}\sum_{<i,j>}\{S_i^2(1-S_j^2) + S_j^2(1-S_i^2)\} \end{aligned} \tag{2}$$

To conserve the particle numbers of the two species, they included the chemical potentials:

$$\begin{aligned} H &= H_S + H_I - \mu_3 N_3 - \mu_4 N_4 \\ &= -J\sum_{<i,j>} S_i S_j - K\sum_{<i,j>} S_i^2 S_j^2 + \Delta\sum_i (S_i^2) \\ &+ h\sum_i S_i + constant \end{aligned} \tag{3}$$

The third term now is a spin splitting term with different energies for the singlet and the doublet. The forth term is in general an external field but doesn't have a physical interpretation here in the context of superfluidity. Their study of this simplified Hamiltonian shows that both superfluidity represented by the Ising interactions (the first term in Eq. 3) and the attractive interactions between atoms of the same species (the second term in Eq. 3) can lead to a phase separation. The interaction intensity ratio J/K will indicate which of the mechanism causes the phase separation. In this work they didn't implement the model with a specific lattice symmetry. Only the nearest neighbour number comes into the splitting energy parameter Δ.

As summarized in [1], the usual computational tools employed to study discrete models are: exact solutions, position space renormalization groups, Monte Carlo simulations and series expansions. In the next few sections I'll summarize some selected work on the model with these different methods.

SOLUTIONS ON 2D HONEYCOMB LATTICE: EXACT SOLUTIONS

Since the birth of BEG model, several exact solutions were found for certain realizations of the model. Here I will discuss the exact solution to Eq. 3(see [4]) on the honeycomb lattice with a constraint:

$$K = -lncoshJ \tag{4}$$

First they analyzed the ground state of this model as $T \to 0$. Now

$$K \to -|J| \tag{5}$$

and the Hamiltonian is:

$$\frac{H}{J} \to \sum_{<i,j>} \left(S_i^2 S_j^2 - S_i S_j\right) + \frac{\Delta}{J}\sum_i S_i^2 - \frac{h}{J}\sum_i S_i \tag{6}$$

for ferromagnetic case ($J > 0$) and

$$\frac{H}{|J|} \to \sum_{<i,j>} \left(S_i^2 S_j^2 + S_i S_j\right) + \frac{\Delta}{|J|}\sum_i S_i^2 - \frac{h}{|J|}\sum_i S_i \tag{7}$$

for antiferromagnetic case ($J < 0$). As expected the ground state for ferromagnetic is one of the 3 ordered states with all spins equal 1,-1, or 0. The ground state for antiferromagnetic case is six-fold degenerate, with the above 3 ordered states and the other 3 antiferromagnetic states in wich nearest neighboring pairs are 1,0,-1,0 and 1,-1.

Now for finite temperature, the constraint Eq. 4 motivated from the high-temperature series expansion now greatly simplifies the problem by the relation:

$$\exp\left(JS_iS_j + KS_i^2S_j^2\right) = 1 + S_iS_j\tanh J \tag{8}$$

now the partition function is:

$$Z = \sum_{S_i}\prod_{<i,j>}(1 + S_iS_jt)\prod_i exp\left(-\Delta S_i^2 + hS_i\right) \tag{9}$$

where $t = \tanh J$. Now we can map this partition function to an eight-vertex model (see [5]), by attaching the nearest neighbor interaction weight t to each occupied bond:

$$Z = Z_8(a, b, c, d) \tag{10}$$

Further applying a weak-graph expansion to the eight-vertex model, and comparing it with the high-temperature expansion for a spin-1/2 Ising model on a honeycomb lattice, they showed the eight-vertex model is equivalent to the Ising model with nearest interaction K_I and an external field H:

$$\begin{aligned} Z_8(a, b, c, d) &= Z_8(a', b', c', d') \\ &= (a'/2coshH)^N (coshK_I)^{-3N/2} Z_I(K_I, H) \end{aligned} \tag{11}$$

where the primed parameters are determined by the weak-graph expansion, and the Ising parameters are:

$$tanhK_I = c'/a' \tag{12}$$

$$tanhH = b'/(a'c')^{\frac{1}{2}} \tag{13}$$

The equivalent Ising model has the same qualitative magnetic behavior as the original BEG model since $K_IJ > 0$. These results follow from the mapping:

1. $J > 0$: ferromagnetic/paramagnetic phase transition boundary is

$$tanhJ < (2 + e^{\Delta})/2\sqrt{3}$$

2. $J < 0$: the Ising model with zero external field is known, and the anti-ferromagnetic/paramagnetic phase transition boundary for the corresponding BEG model is

$$e^{2|K_I|} = 2 + \sqrt{3}$$

SOLUTIONS ON 2D SQUARE LATTICE: POSITION-SPACE RENORMALIZATION GROUP

Another analytical method to study the BEG model is the position-space renormalization group. In [3], they focused on the ferromagnetic and attractive biquadratic interaction case, i.e., $J, K > 0$. Besides the square lattice symmetry and the up-down symmetry, there is a Griffiths symmetry intrinsic to the BEG model which maps it to an Ising model when $\Delta \ll -1$ or $J = 0$. They conducted a certain coarse graining while preserving the above symmetries:
(1) Group sites into cells. The cells are now every other plaquette of the original square lattice, and now they still form a square lattice with a doubled lattice constant, thus the scaling factor later will be $b = 2$.
(2) Determine the cell spin with a double majority rule. The majority rule is: $S_{cell}^{\alpha} = sgn(\sum_i S_i^{\alpha})$. For cells with 2 up spins and 2 down spins, the coarse grained partition function takes the cell to have 1/2 weight of spin-up and 1/2 weight of spin-down. The double majority rule is:

1. Reduce all site BEG spins $s = 0, \pm 1$ to the associated Ising spin $t = 2s^2 - 1 = \pm 1$.

2. Generate the associated Ising spins for the cell $t'^{\alpha} = 2(s'^{\alpha})^2 - 1$ by the majority rule applied on t_i^{α}. In the 'non-magnetic' case ($t' = -1$) the cell BEG spin is determined $s' = 0$.

3. Determine the cell BEG spin s' in the 'magnetic' case ($t' = 1$) by applying the majority rule directly to the site BEG spins s_i^{α}. Notice now sgn$(\sum_i S_i^{\alpha}) \neq 0$.

In addition to this scheme, they introduced a modification to the majority rule by assigning a proportion v of these configurations such as 1,1,1,-1, which in the above only contribute to the cell spin $s' = 1$, to the cell spin $s' = 0$. v is then best fitted to -0.06453 by yielding the correct critical interaction $J = \frac{1}{2} ln(\sqrt{2} + 1)$ for the Ising model.

After the coarse graining, they conducted renormalization group analysis with the scaling factor $b = 2$, and find 13 fixed points in the parameter space. The subsequent renormalization group analysis are the standard procedure as described in [1].

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