

Renormalization of the Velocity Anisotropy due to Coulomb Interaction in 3D Semimetals

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We consider anisotropic (3+1)D semimetals with linear electronic spectrum. Our purpose is to investigate the influence of instantaneous Coulomb interaction on the anisotropy of the velocity. We consider both weak and strong coupling limits. To analyze the system, we construct renormalization group (RG) equations for the components of the velocity. We show that in the weak coupling limit Coulomb interaction reduces initial anisotropy of the velocity. In the strong coupling limit, initial anisotropy remains unchanged in the leading order.

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Introduction. — After the experimental realization of graphene, this material became very interesting and popular from both experimental and theoretical point of view. Near the Dirac points, electrons in graphene are described by the relativistic Dirac equation, and thus have linear spectrum. This fact makes the theory of graphene somewhat similar to the Quantum Electrodynamics (QED).

Another interesting example of systems with linear electronic spectrum is (3+1)D Dirac semimetals [2–4]. These systems are even more similar to the conventional (3+1)D QED. These are the systems we consider in this work.

However, despite many common features, there is significant difference between QED and condensed matter systems with linear spectrum. In graphene and other semimetals, the Fermi velocity is much smaller than the speed of light, $v_F \ll c$. This is the reason why the bare Coulomb interaction can be considered as a static instantaneous interaction.

Another remarkable consequence of 'low' Fermi velocity is that fine-structure constant $\alpha_0 = e^2/\hbar v_F$, which serves as a small parameter for the perturbation theory in QED, can be of the order 1 in semimetals (in graphene, for example, $\alpha_0 \approx 2.17$). However, in condensed matter systems, one need to take into account permittivity ε of the material, and thus effective fine-structure constant $\alpha = e^2/\varepsilon\hbar v_F$ can be still a small number.

In studying semimetals, the one of the important questions is how the long-range Coulomb interaction renormalizes the Fermi velocity. This question has been considered in [1]. It was shown that in the weak-coupling limit $\alpha \ll 1$ energy spectrum is renormalized in the following way:

$$\omega(k) = \pm v_F k \left(1 + \frac{2e^2}{3\pi v_F \varepsilon} \ln(k_{\max}/k) \right), \quad (1)$$

where k_{\max} is the ultraviolet cutoff, which is of the order of the period of the reciprocal lattice, and we put $\hbar = 1$ from now on. It is clear from (1), that the Coulomb interaction increases Fermi velocity, and fine-structure constant α becomes even smaller. Alternatively, RG equation for α has the form

$$\frac{d\alpha}{d \ln b} = -\frac{2}{3\pi} \alpha^2, \quad (2)$$

with $b = k_{\max}/k$. Thus, $\alpha = 0$ is a stable fixed point.

Another interesting problem is to find the RG flow in the anisotropic (3+1)D Fermi semimetals [3, 4]. This is the question we consider in this work. Specifically, we are interested in the renormalization of the ratio v_1/v_2 , where v_1 and v_2 are in-plane and out-of-plane components of Fermi velocity. We consider both the weak-coupling and strong-coupling limits.

In the weak-coupling limit, we derived explicit RG equations for components of the velocity in one-loop approximation and showed that the Coulomb interaction reduces the initial anisotropy.

In the strong-coupling limit, to derive RG equations is much more subtle problem. Recent result showed that one should expect critical point in this limit [5]. Unfortunately, more work need to be done in order to derive RG equations in that limit. However, we have managed to show that in this case the problem can be reduced to the isotropic case, and the Coulomb interaction does not change the ratio v_1/v_2 in the leading order.

The model. — We follow the idea of [1] and consider a system of (3+1)D two-component massless Dirac fermions interacting through instantaneous Coulomb interaction. The action has the form (with the partition function $Z = \int D\psi^+ D\psi D\phi e^{iS}$)

$$S = \int dt \left\{ \sum_j \int d^3\mathbf{r} \psi_j^\dagger(\mathbf{r}) \left(i \frac{\partial}{\partial t} + i v_1 \sigma_x \frac{\partial}{\partial x} + i v_1 \sigma_y \frac{\partial}{\partial y} + i v_2 \sigma_z \frac{\partial}{\partial z} + \phi(\mathbf{r}) \right) \psi_j(\mathbf{r}) + \frac{1}{2} \int \frac{d^3\mathbf{k}}{(2\pi)^3} \frac{\varepsilon k^2}{4\pi e^2} |\phi(\mathbf{k})|^2 \right\} \quad (3)$$

where ψ_j are two-component spinors of different flavors, σ_i are Pauli matrices, v_1 and v_2 are components of the Fermi velocity, ε is the permittivity of material, and ϕ stands for scalar Coulomb field. In this notations, casual Green functions in the Fourier space have the form:

$$\begin{aligned} G_{ij}(\omega, \mathbf{k}) &= -i \langle \psi_i(\omega, \mathbf{k}) \psi_j^\dagger(\omega, \mathbf{k}) \rangle = \\ &= \frac{\delta_{ij}}{\omega - v_1(\sigma_x k_x + \sigma_y k_y) - v_2 \sigma_z k_z + i \varepsilon \text{sign} \omega} = \\ &= \frac{\omega + v_1(\sigma_x k_x + \sigma_y k_y) + v_2 \sigma_z k_z}{\omega^2 - v_1^2(k_x^2 + k_y^2) - v_2^2 k_z^2} \delta_{ij}, \end{aligned} \quad (4)$$

$$V_0(\omega, \mathbf{k}) = -i \langle |\phi(\omega, \mathbf{k})|^2 \rangle = \frac{4\pi e^2}{\varepsilon k^2}. \quad (5)$$

However, we will use screened Coulomb interaction in the random phase approximation (RPA) instead of the bare Coulomb interaction V_0 . From the Dyson equation $V = V_0 + V_0 \Pi V$, one has

$$V(\omega, \mathbf{k}) = \frac{4\pi e^2}{\varepsilon k^2 - 4\pi e^2 \Pi(\omega, \mathbf{k})}, \quad (6)$$

where the polarization operator $\Pi(\omega, \mathbf{k})$ is given by (see Fig. 1)

$$\Pi(\omega, \mathbf{k}) = -iN \int \text{Tr} G_{jj}(\Omega, \mathbf{q}) G_{jj}(\omega + \Omega, \mathbf{k} + \mathbf{q}) \frac{d\Omega d^3\mathbf{q}}{(2\pi)^4}, \quad (7)$$

where N is the number of electron flavors and there is no summation over j . The minus sign comes from the fermionic loop. One can easily see, that simple rescaling $k_z \rightarrow v_1 k_z / v_2$, $q_z \rightarrow v_1 q_z / v_2$ makes the integral isotropic,

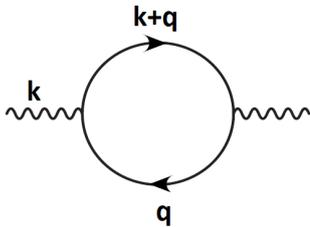


FIG. 1: Polarization operator $\Pi(\omega, \mathbf{k})$ in the one-loop approximation.

and the polarization operator can be found:

$$\Pi(\omega, \mathbf{k}) = -\frac{N \tilde{k}^2}{12\pi^2 v_2} \ln \frac{v_1 k_{\text{max}}}{\sqrt{v_1^2 \tilde{k}^2 - \omega^2}}, \quad (8)$$

where $\tilde{\mathbf{k}} = \{k_x, k_y, v_2 k_z / v_1\}$. Though the calculation of $\Pi(\omega, \mathbf{k})$ is rather cumbersome, it is standard, and we do not present it here. Good examples of the calculation of integrals with loops can be found, for example, in [6].

Result (8) is the straightforward generalization of the result obtained in [1]. As we see, in the case of small fine-structure constant and not so big N , screening becomes small and one can use bare Coulomb interaction, $V \approx V_0$. However, the situation becomes totally different in the large-coupling limit. In this case, the bare interaction is totally screened, and one ends up with what is known as universal limit, $V \approx -1/\Pi$. In this limit, the effective interaction does not depend on the initial long-range interaction at all. This situation can be realized via either big fine-structure constant or large number of flavors N . The last case is known as $1/N$ expansion [7]. In this case, one can express the answer as a series in powers of $1/N$.

In order to calculate the renormalization of the velocity, we calculate the self-energy part in the one-loop approximation (see Fig. 2):

$$\Sigma(\omega, \mathbf{k}) = i \int V(-\Omega, -\mathbf{q}) G(\omega + \Omega, \mathbf{k} + \mathbf{q}) \frac{d^3\mathbf{q} d\Omega}{(2\pi)^4}. \quad (9)$$

In order to avoid strong ultraviolet divergence, we will calculate $\delta\Sigma(\omega, \mathbf{k}) = \Sigma(\omega, \mathbf{k}) - \Sigma(0, 0)$. $\Sigma(0, 0)$ is a constant which can be treated as a shift of chemical potential. Thus, it does not affect the renormalization of the velocity.

The weak-coupling limit. — We consider the weak-coupling limit first. In this case, $V(\omega, \mathbf{k}) = 4\pi e^2 / \varepsilon k^2$.

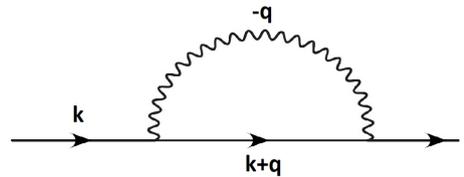


FIG. 2: The electron self-energy $\Sigma(\omega, \mathbf{k})$ in the one-loop approximation.

Despite the calculation of the integral (9) is quite cumbersome, it is straightforward. For small energies and

momenta we have the result

$$\delta\Sigma(\omega, \mathbf{k}) = \frac{e^2}{2\pi\varepsilon} \left\{ \left(\frac{\sqrt{a}}{a-1} + \frac{\sqrt{a}(a-2)\arctan\sqrt{a-1}}{(a-1)^{3/2}} \right) (k_x\sigma_x + k_y\sigma_y) + \left(-\frac{2}{a-1} + \frac{2a\arctan\sqrt{a-1}}{(a-1)^{3/2}} \right) k_z\sigma_z \right\} \ln \frac{k_{\max}}{k}, \quad (10)$$

for $a = v_1^2/v_2^2 > 1$, and

$$\delta\Sigma(\omega, \mathbf{k}) = \frac{e^2}{2\pi\varepsilon} \left\{ \left(-\frac{\sqrt{a}}{1-a} + \frac{\sqrt{a}(2-a)\operatorname{arctanh}\sqrt{1-a}}{(1-a)^{3/2}} \right) (k_x\sigma_x + k_y\sigma_y) + \left(\frac{2}{1-a} - \frac{2a\arctan\sqrt{1-a}}{(1-a)^{3/2}} \right) k_z\sigma_z \right\} \ln \frac{k_{\max}}{k}, \quad (11)$$

for $a = v_1^2/v_2^2 < 1$.

To find the renormalization of the spectrum, we use the Dyson equation in the form $G_0^{-1} = G^{-1} + \Sigma$. The spectrum is determined by the poles of the Green function G . We see that interaction keeps the spectrum nearly linear and just renormalizes the velocity (at least, this is correct for not exponentially small momenta, $k > k_{\max} \exp(-2\pi/\alpha)$). In the isotropic limit $a \rightarrow 1$, both formulas (10) and (11) reproduce the result (1).

In order to proceed further, let us write RG equations for the components of velocity. For that purpose, we need to integrate self-energy part in the shell from k_{\max}/b to k_{\max} . This leads to the following RG equations

$$\begin{cases} \frac{dv_1}{d\ln b} = \frac{e^2}{2\pi\varepsilon} \left(\frac{\sqrt{a}}{a-1} + \frac{\sqrt{a}(a-2)\arctan\sqrt{a-1}}{(a-1)^{3/2}} \right) \\ \frac{dv_2}{d\ln b} = \frac{e^2}{2\pi\varepsilon} \left(-\frac{2}{a-1} + \frac{2a\arctan\sqrt{a-1}}{(a-1)^{3/2}} \right) \end{cases} \quad (12)$$

for $a > 1$, and

$$\begin{cases} \frac{dv_1}{d\ln b} = \frac{e^2}{2\pi\varepsilon} \left(-\frac{\sqrt{a}}{1-a} + \frac{\sqrt{a}(2-a)\operatorname{arctanh}\sqrt{1-a}}{(1-a)^{3/2}} \right) \\ \frac{dv_2}{d\ln b} = \frac{e^2}{2\pi\varepsilon} \left(\frac{2}{1-a} - \frac{2a\operatorname{arctanh}\sqrt{1-a}}{(1-a)^{3/2}} \right) \end{cases} \quad (13)$$

for $a < 1$. One can reformulate these equations in terms of the running coupling constants $v_1(q)$ and $v_2(q)$. This approach means that we consider v_1 and v_2 at the momentum scale q . All one need to do is just to substitute $d\ln b = -d\ln q$ in (12) and (13).

It can be easily derived analytically, that in both cases $a > 1$ and $a < 1$ the ratio v_1/v_2 goes to 1. This statement is confirmed by the numerical calculation (see Fig. 3). Thus, the Coulomb interaction reduces the anisotropy in the weak-coupling limit in (3+1)D.

The strong-coupling limit. — To calculate the RG equations in the strong-coupling limit is much more difficult problem. Unlike (2+1)D case [7], in (3+1)D polarization operator $\Pi(\omega, \mathbf{k})$ contains the ultraviolet cutoff

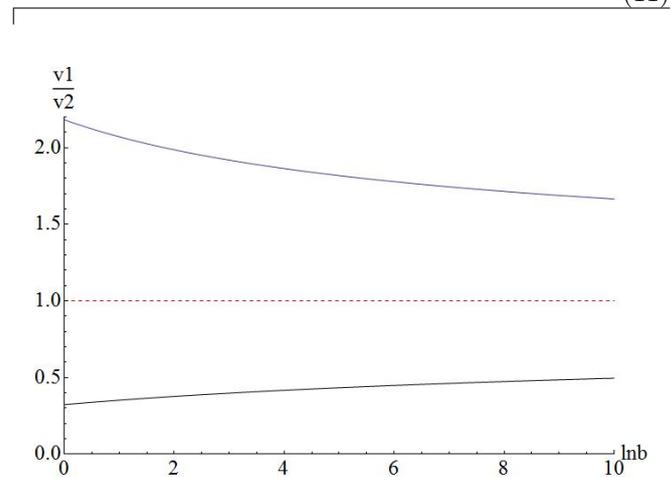


FIG. 3: RG flow of the ratio v_1/v_2 in the weak-coupling limit $e^2/2\pi\varepsilon = 0.3$. Blue line corresponds to the initial velocities $v_1 = 2.4$, $v_2 = 1.1$. Black line corresponds to the initial velocities $v_1 = 1.1$, $v_2 = 3.4$. In both cases, the Coulomb interaction reduces the anisotropy.

k_{\max} explicitly (see (8)). This fact makes it more difficult to construct RG equations.

However, we can find the anisotropy flow in the leading $1/N$ order even without RG equations. In order to do that, one need to remind that in the strong-coupling limit effective interaction is given by the polarization operator only, $V(\omega, \mathbf{k}) = -1/\Pi(\omega, \mathbf{k})$. The polarization operator is determined by the electron Green functions (see (7)). That means that eventually self-energy part $\Sigma(\omega, \mathbf{k})$ is determined by electron Green functions only. Now, one can simply rescale all of the z -components of the momenta in all Green functions and to reduce the calculation of Σ to the isotropic case. As a consequence, interaction does not change the ratio v_1/v_2 in the leading order in $1/N$ expansion.

Our result agrees with the calculation for anisotropic graphene [8]. However, it was shown in [8] that next-order terms in $1/N$ expansion lead to the reduction of the anisotropy in (2+1)D case. That is what we expect for (3+1)D case too, as this result agrees with our calculation for the weak-coupling limit. Nevertheless, this statement need to be carefully verified in the future.

Conclusion. — In this work we investigated the influence of the Coulomb interaction on the anisotropy in the 3D Dirac semimetals. We considered both weak- and strong-coupling limits. In the weak-coupling limit, we constructed RG equations and showed that the Coulomb interaction reduces the initial anisotropy of the components of the velocity. In the strong-coupling limit, we showed that in the leading order in $1/N$ expansion, the interaction does not change the initial anisotropy.

As a possible extension of this work, we plan to explore strong-coupling limit in detail. Specifically, we plan to construct RG equations for that case to the higher or-

der terms in $1/N$ expansion. It will allow us to understand whether interaction reduces or increases the initial anisotropy in the strong-coupling limit.

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- [1] A. A. Abrikosov, S. D. Beneslavskii, *Sov. Phys. JETP* **32**, 699 (1971).
 - [2] Z. Wang *et al.*, *Phys. Rev. B* **85**, 195320 (2012).
 - [3] Z. Wang *et al.*, *Phys. Rev. B* **88**, 125427 (2013).
 - [4] M. Neupane *et al.*, *Nat. Comm.* **5**, 3786 (2014).
 - [5] J. González, arXiv:1402.6992
 - [6] M. E. Peskin, D. V. Schroeder, *An Introduction to Quantum Field Theory* (Perseus Book, New York, 1995).
 - [7] D. T. Son, *Phys. Rev. B* **75**, 235423 (2007).
 - [8] Jing-Rong Wang and Guo-Zhu Liu, *Phys. Rev. B* **89**, 195404 (2014).