A Survey of Percolation Theory and its Applications

Alexandru Bacanu[1]

[1]Department of Physics, Massachusetts Institute of Technology, Cambridge MA, 02138

bacanua@mit.edu

Abstract

Given a lattice and an occupation probability p, there exists a critical probability p_c, at which clusters of occupied lattices span the entire system. In this analysis, we investigate the critical probability of various lattices in one, two, and infinite dimensions through simulations and renormalization calculations. We present numerical estimates for p_c as well as the ratio of critical exponents β/ν and γ/ν on the two-dimensional square lattice. We also describe a game of "odd chess" where the lattice is not an alternating black-white arrangement, but rather, a random lattice, and discuss how the probability of checkmate changes as a function of the occupation probability.

I. Introduction

Percolation theory applies to many disparate physical phenomena such as polymeric gelation, crystalline impurities, as well as disease propagation through an orchard. One appealing aspect of percolation theories is their universality [1], in that their behavior depends only upon the spatial dimension close to criticality and thus, different systems display similar behavior. In this paper, we discuss both site and bond percolation, but it should be noted that site percolation is more general than bond percolation because all site percolation models can be mapped to bond percolation models, but the inverse statement is not true [2]. We shall now define some necessary terminology in Table 1, so that the following discussion proceeds as smoothly as possible.

Table 1

Lattices in Two Dimensions

Observable	Definition	Exponent
S-mers per Lattice Pt	$n_s(p) = \frac{S-mers}{L^2}$	
Gel Fraction	$P_{gel} + P_{sol} = 1$	β
Center of Mass (com)	$\vec{r}_{com} = \frac{1}{N}\sum_{i=1}^{N} \vec{r}_i$	
Radius of Gyration	$R_g^2 = \frac{1}{N}\sum_{i=1}^{N} \lvert\vec{r}_i - \vec{r}_{com}\rvert^2$	
Correlation Length	$\xi^2 = \frac{\sum_s 2R_s s^2 n_s(p)}{\sum_s s^2 n_s(p)}$	ν
Mean Cluster Size	$\sum_s s^2 n_s(p)$	γ

II. Percolation in 1 Dimension

We begin with a treatment of percolation in one dimension as well as infinite dimensions, where we may obtain exact results. Consider an infinite one dimensional chain of bonds (or sites). Unless $p = 1$, there can be no percolation; only finite sized molecules will ever be created. However, it is interesting to quantify the distribution of lengths for various p. This analysis can be applied to linear condensation polymers [3]. An N-mer is a polymer with N monomers, meaning it has $N-1$ reacted and 2 unreacted bonds. Thus, the number of N-mers per monomer becomes

$$n(p, N) = p^{N-1}(1-p)^2 \tag{1}$$

From this we may extract the number average degree of polymerization, N_n as $\frac{1}{\sum_{i=1}^{\infty} n(p,N)} = \frac{1}{p-1} = N_n$, thus confirming the existence of a critical point at $p = 1$. Using Eq. 1, we find the weight fraction of N-mers to be $w_N(p) = Np^{N-1}(1-p)^2$. Thus, the weight fraction distribution may be approximated with an exponential cutoff when $p \approx 1$

$$w_N(p) \approx \left(\frac{N}{N_n^2}\right) \exp\left(\frac{-N}{N_n}\right) \tag{2}$$

Though the one dimensional case lacks an effective critical point with $p_c < 1$, it serves to illustrate ideas that recur in higher dimensional models, such as exponential cutoff functions, as well as weight distributions, and it still models certain real-world scenarios, such as the telegraph game [4] .

III. Mean Field Theories (D = ∞)

Consider the Cayley tree shown in below in Figure 1, and assume that f bonds emanate from each atom. Each atom has $f-1$ potential children, and assume that additional bonds form at the terminus of the tree with probability p so that each atom has, on average, $p(f-1)$ children.

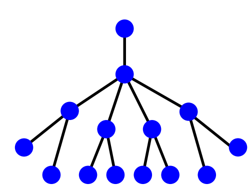


Figure 1: *The Cayley tree has an exponential branching structure and is unable to fit into spaces of any finite dimension.*

Thus, for infinite trees, we must have

$$p_c = \frac{1}{f-1} \tag{3}$$

Using the binomial theorem, we derive the number fraction of N-mers.

$$n_{N,p} = \frac{[(f-1)N]!}{N![(f-2)N+1]!} p^{N-1}(1-p)^{(f-2)N+1} \tag{4}$$

Since every N-mer consists of $N-1$ reacted bonds and $(f-2)N+1$ unreacted bonds. Close to the gel point, expanding in terms of our order parameter, $\epsilon = p/p_c - 1$ we obtain

$$n_{N,p} \approx \sqrt{\frac{f-1}{2\pi(f-2)}} N^{-3/2} \exp\left(-\frac{1}{2}\frac{(f-1)}{(f-2)}\epsilon^2 N\right) \tag{5}$$

from which we extract our first critical exponent from the characteristic polymer length, $N^* \approx \epsilon^2$, thus $\sigma = \frac{1}{2}$. The gel fraction is zero below the critical point and increases as a power law right above p_c. Letting Q be the probability that a randomly selected monomer is not connect to the gel, we find

$$Q = 1 - p + Q^{f-1} \tag{6}$$

since the site could be unreacted with probability $1-p$ or reacted, but none of the $f-1$ bonds are attached to the gel. From this, close to the gel point, we may approximate Eq 4 as $P_{gel} \approx (p - p_c)$ and thus, our second critical exponent becomes $\beta = 1$. Defining the k^{th} moment of the number fraction distribution as

$$m_k = \sum_{N=1}^{\infty} N^k n_{p,N} \tag{7}$$

From this, close to the critical point, we can find the weight average degree of polymerization $\frac{m_2}{m_1} \approx \epsilon^{-1}$ and we obtain $\gamma = 1$.

The Cayley tree does not fit into space of any dimensions due to the exponential increase in the number of monomers due to branching. However, Takashi and Slade [5] have shown that the upper critical dimension is $d_c = 6$ above which the mean field theory describes all lattice percolations.

IV. Finite Size Scaling

One problem that presents itself is that on a finite lattice, p_c is not well defined, because percolation can occur with non-zero probability p^L for any p. Thus, all exponents are only well defined in the limit $L \to \infty$. There are no true singularities on a finite lattice and two length scales, ξ and L define the system. If $\xi < L$, the finiteness of the system is unimportant. However, if $L \approx \xi$, a new behavior emerges because at this point, the whole lattice becomes occupied. $P(p) = |p - p_c|^\beta = \xi^{-\beta/\nu}$ and thus, close to criticality $P(p, L) \approx L^{-\beta/\nu}$ We guess a scaling ansatz of the form

$$P(p, L) = \xi^{-\beta/\nu} F_L(\xi/L) \tag{8}$$

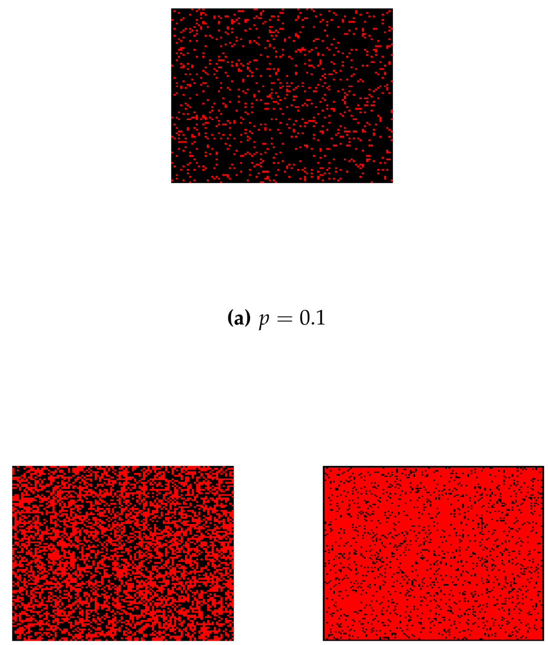
where

$$F_L(\xi/L) = \begin{cases} (L/\xi)^{-\beta/\nu} & : \xi/L >> 1 \\ constant & : \xi/L << 1 \end{cases}$$

This analysis allows us to estimate the critical exponents by plotting values of observables for different values of L as will be further discussed in the following section.

V. Simulations

Critical exponents can be deduced from simulations on various lattices. Empty lattices are created and each point is filled with probability p. Percolation is defined to occur whenever a chain of contiguous occupied sites spans the entire lattice. In Figure 2, we can see that at $p = 0.1$, the filled lattice points (in red) begin as small islands, at $p = 0.6$, the filled points percolate through the system, and at $p = 0.9$ the empty lattice points only fill small islands in the lattice.



(a) $p = 0.1$

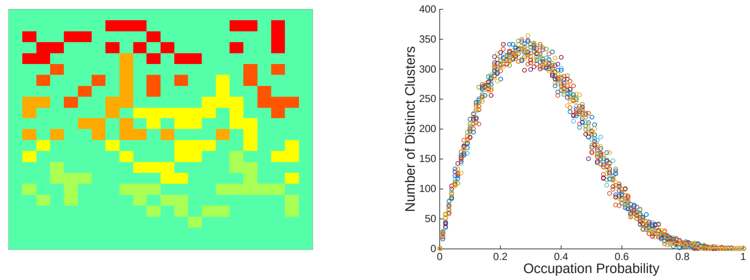
(b) $p = 0.6$

(c) $p = 0.9$

Figure 2: *As p increases, the clusters become larger, until they span the entire system.*

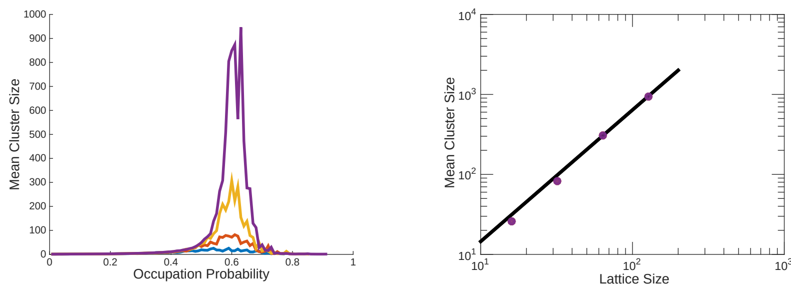
The Hoshen-Kopelman algorithm [6] was used to identify distinct clusters in the lattice. Each time the algorithm encounters a point not connected to any previous clusters, it is labeled as a new cluster. If a lattice point connects two clusters previously thought to be distinct, they are both relabeled to include the new information.

At the critical threshold, it is not immediately clear which observables diverge and which do not.



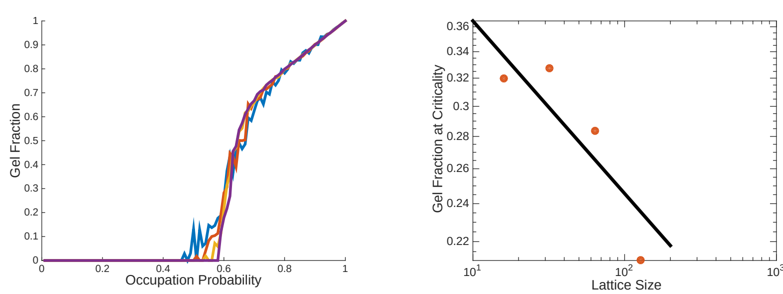
(a) (b)

Figure 3: *(a): Hoshen-Kopelman cluster finding algorithm. Each color represents a distinct cluster. (b): Maximum cluster number does not occur at* $p_c = 0.592$.



(a) (b)

Figure 4: *(a): Cluster size distribution for L = 16, 32, 64, 128. By seeing where the maximum of each distribution occurs, we estimate* $p_c = 0.6075$*, which is not far away from the accepted value of* $p_c = 0.592$*. (b): We obtain the exponent* $\gamma/\nu \approx 1.64$.



(a) (b)

Figure 5: *(a): Gel Fraction for L = 16 (blue), 32 (red), 64 (yellow) 128 (purple). We see that fluctuations decrease as L grows larger (b): We obtain the exponent* $\beta/\nu \approx 0.17$.

We can see above in Figure 3 that the number of clusters does not diverge at the critical point. In fact, the maximum number is reached around $p \approx 0.3$. However, other observables, such as the correlation length and the mean cluster size do diverge at the critical point as can be seen in Figures 4 and 5. Using the scaling relations $\beta = \frac{\tau-2}{\sigma}$, $\gamma = \frac{3-\tau}{\sigma}$, $\tau = \frac{d}{d_f} + 1$, where d and d_f are the spatial and fractal dimension, respectively, we obtain estimates for the critical exponents and compare them against their standard values in Table 2.

Table 2

Square Lattice in Two Dimensions		
Exponent	Prediction	Actual
p_c	0.608	0.592
γ/ν	1.644	1.792
β/ν	0.172	0.104
τ	2.095	2.055
d_f	1.827	1.896

Table 3: *Most values are within 10 percent of their accepted value, except the value β/ν, where it was difficult to obtain an accurate estimate from the sparse data.*

VI. Renormalization Group in Two Dimensions

I. Site Percolation on the Triangular Lattice

We may construct a renormalization treatment for the triangular lattice in two dimensions which will yield information about the system close to criticality. [7]

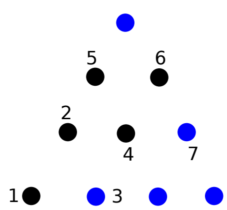


Figure 6: *There are two ways for a grouping of three spins to be occupied: Either two spin are occupied as in triangle 1-2-3, or all three spins are occupied, as in triangle 4-5-6.*

We shall renormalize by considering groupings of three spins as in Figure 6. Define each to be "occupied" if there are at least two bonds (ie, the triangle is traversed). This can be seen by considering triangle 1-2-3 or triangle 4-5-6, where both are considered to be occupied. If the probability of creating a bond is p, then

$$P(occupied) = p^3 + 3p^2(1-p) \tag{9}$$

is the probability that a grouping is occupied. At the fixed point p_c, the grouping probability is mapped upon itself, and we obtain $p_c = 0.5$. To obtain ν, we use

$$\nu = \frac{\log b}{\log\left(\frac{dP}{dp}\right)|_{p=p_c}} = \frac{\log\left(\sqrt{3}\right)}{\log\left(6p_c - 6p_c^2\right)} \approx 1.355 \tag{10}$$

compared to an exact value of $\nu = 4/3$.

II. Bond Percolation on the Square Lattice

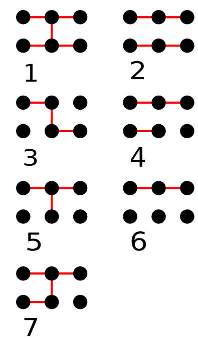


Figure 7: *Seven ways to span a renormalization subsection of a square lattice.*

From Figure 7 above,
P(occupied) = $p^5 + p^4(1-p) + 2p^3(1-p)^2 + 4p^3(1-p)^2 +$

$$2p^3(1-p)^2 + 2p^2(1-p)^3 + 4p^4(1-p) \tag{11}$$

At the fixed point p_c, the grouping probability is mapped upon itself, and we obtain $p_c = 0.5$. We obtain ν in a similar manner as in the triangular lattice

$$\nu = \frac{\log 2}{\log\left(\frac{dP}{dp}\right)|_{p=p_c}} = 1.428 \tag{12}$$

compared to an exact value of $\nu = 4/3$

We see that in both cases, we are able to obtain exact values for p_c and close approximations for the critical exponents.

VII. Odd Chess

Consider a standard 8×8 random square lattice with occupation probability p with occupied points painted black and unoccupied points painted white. We now modify how pieces can move to take into account to altered version of the lattice, and for simplicity, we consider only kings and queens.
1) Queens beginning on black lattice points can move (and

thus check/checkmate) to any other black lattice point connected to the same cluster. The same is true for queens starting on white lattice points.
3) Kings can move as in standard chess, hopping one square in any direction, regardless of whether it is a black or white square.
4) Player one's king can be adjacent to player two's king (and thus checkmate player two's king) so long as it is guarded by a queen in the same cluster. If not, player one's king can capture player two's king. Player two's king can thus be checkmated regardless of whether player one's king is on a white or black square.
Notice that the king is the only piece not confined to move in the cluster it begins in. In standard chess, if player one has a king and a queen, player one is always able to checkmate player two (assuming no blunders are made). However, in "odd chess", this is not so. Consider an 8×8 lattice with $p = 0.5$. In this case, it is quite probable that neither white nor black clusters span the lattice, and thus player two's king can hop to a lattice point unable to be reached by player 1's queen. Thus, unlike standard chess, "odd chess" is not always a well defined game. We now consider a 2×2 lattice.

I. 2×2 lattice

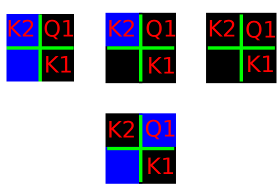


Figure 8: *In the three scenarios in the first row, player one checkmates player two. In the last scenario, there is no checkmate. The colors blue and black are used for greater visibility.*

The only scenario in which checkmate cannot happen on a 2×2 lattice is when there are two black and two white lattices, each on the diagonal. This can be seen in the bottom part of Figure 8. Thus,

$$P(checkmate) = 1 - 2p^2(1-p)^2 \tag{13}$$

Thus the game is defined in excess of 87.5 percent of the times on the 2×2 lattice.

VIII. 3×3 lattice

The 3×3 lattice is more complicated because we must consider the initial configuration of the system as well as any possible moves player two has.

$$P(checkmate) = p^9 + (1-p)^9 + \frac{76}{9}\left(p(1-p)^8 + p^8(1-p)\right) + \ldots \tag{14}$$

Even for a simple 3×3 lattice, the combinatorics become quite complicated for more than one white (or black) square. However, it is important to note that finite size edge effects are quite important in chess; one simple way to checkmate an opponent is cornering their king in a corner or on the edge of the board. Chess on an infinite board would be quite boring. No checkmate would happen until $p \geq p_c$. In fact for $(1 - p_c) \leq p \leq p_c$, one would require a countably infinite number of queens to checkmate an opponent's king because of the lack of percolation by either black or white squares.
We end with a simple discussion of the "movement entropy" associated with different pieces.
1) The king can explore all squares. $S_{king} \propto \ln\left(L^2\right)$
2) The queen can only explore her cluster, thus, if we place her randomly, her entropy is proportional to the mean cluster size. $S_{queen} \propto \ln\left((p - p_c)^{-\gamma}\right)$

IX. Discussion

Percolation theory is a rich and deeply varied subject with many application from polymer gelation to disease propagation. We have reviewed a wide segment of the theory, spanning the analytical results obtainable from percolation in one dimension as well as on the Cayley lattice (infinite dimensional). Simulations of percolation on the square lattice revealed rich structure as finite size scaling needed to be used to extract information from the "pseudo-critical point" (as true phase transitions occur only in infinite systems). Renormalization group techniques provided us with exact values for p_c and very close agreement with the accepted result for ν. A discussion of "odd chess" revealed that even for seemingly trivial 2×2 and 3×3 lattices, the game is not always well defined, and infinite numbers of queens are needed to checkmate a king for $(1 - p_c) \leq p \leq p_c$ on an infinite lattice. It would be interesting to generalize the above discussion to larger lattice sizes, especially 8×8 lattices, but as mentioned finite lattice edge effects are especially important in chess in order for a checkmate to occur. It would also be interesting to try to quantify the edge effect, and to see how the probability of checkmate decreases as $L \to \infty$ and $p < p_c$.

References

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