Renormalization Group Analysis of Kuramoto's Model

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Kuramoto's model is an important model that describes synchronization phenomena occurring from the coupling between a system of oscillators. Here, we review the original globally coupled Kuramoto model and introduce two renormalization group approaches to the nearest neighbor version on a regular lattice. We then attempt to apply these techniques to small-world networks which lie between regular and random networks, and compare our results with numerical simulations.

I. INTRODUCTION

Synchronization is a natural phenomena which can be found in a variety of different systems from synchronized flashing in fireflies [1] and social dynamics [2, 3] to collective behavior in Josephson junction arrays [4], chemical oscillators [5] and optomechanical arrays [6]. The wide variety of systems in which such a synchronization transition can occur demands for a unifying mathematical framework that can capture universal features shared by these systems, and it is also interesting to understand more fundamentally the underlying mechanisms and limitations of synchronization.

An important step towards a theory of synchronization was the work by Kuramoto [7] which provided an exactly solvable system that exemplified the dynamics of the system, displaying a phase transition. The stability problem of the various branches was analyzed rigorously by adding in stochastic noise and considering the continuum limit of the model [8].

In this paper, we shall be focusing on nearest neighbor coupled Kuramoto models and discuss how ideas from statistical mechanics such as renormalization can be applied to the analysis. In particular, we shall consider a general RG argument which gives a lower bound to the lower critical dimension and a local decimation procedure in 1D. We then examine renormalization approaches on a small-world network and discuss scaling behavior of the system.

II. ORIGINAL KURAMOTO MODEL AND RELATED WORK

To capture the fundamental synchronization features of a system, it is natural to abstract out most of the degrees of freedom and retain only a phase which describes the location of each oscillator on its unperturbed limit cycle.

The original model formulated by Kuramoto [7] included a uniform all-to-all sinusoidal coupling between N such phase oscillators, with the time derivative of the ith oscillator's phase given by

$$\frac{d\theta_i}{dt} = \omega_i + K\sum_{j=1}^{N}\sin(\theta_j - \theta_i), \tag{1}$$

where ω_i is the natural frequency of the ith oscillator drawn from a distribution $g(\omega)$ and K is the coupling constant. The sum runs over all other oscillators in the system, indicating a global coupling. We also note here that the sinusoidal coupling can be viewed as the dominant term in a Fourier expansion of the interactions, which is already sufficient to capture the most prominent features of the synchronization transition.

To solve the steady state of this equation and understand the synchronization properties, we define an order parameter [9]

$$z = re^{i\psi} = \frac{1}{N}\sum_{j=1}^{N} e^{i\theta_j}. \tag{2}$$

Multiplying this by $e^{-i\theta_i}$, redefining phases so that $\psi = 0$ and plugging back into the original Kuramoto's model Eq. (1) gives us a mean field equation

$$\dot{\theta}_i = \omega_i + Kr\sin(\psi - \theta_i), \tag{3}$$

which allows us to identify a population of drifting oscillators with $|\omega_i| > Kr$ and stably locked-in oscillators with $\omega = Kr\sin\theta$. After averaging, the order parameter will only receive contributions from the locked-in oscillators, hence we have the self-consistency condition for the order parameter:

$$\begin{aligned} r = \langle e^{i\theta}\rangle_{lock} &= \int_{-Kr}^{Kr} e^{i\theta(\omega)} g(\omega) d\omega \\ &= Kr\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^2\theta\, g(Kr\sin\theta) d\theta \end{aligned} \tag{4}$$

This equation has two solutions: an unsynchronized state with $r = 0$, and a partially synchronized branch that bifurcates out at $K_c = \frac{2}{\pi g(0)}$, see Fig. 1 for a schematic of the phase diagram. We thus see that synchronization occurs naturally in this model, and the critical scaling of the order parameter with respect to the coupling constant can in principle be calculated by taking the derivative of the self-consistency equation.

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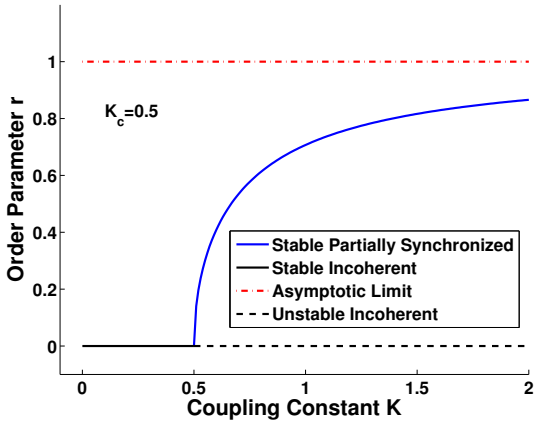

FIG. 1. Phase diagram for the original kuramoto.

To determine the stability of the various branches, one can go into the continuum limit of the mean-field Kuramoto equation Eq. 3. With the addition of perturbative noise, this will then correspond to a Fokker-Planck equation [8]. Recently, Ott and Antonsen [10] have also developed an ansatz which allows one to reduce the differential-integral equations to the low-dimensional characteristics of the order parameter only, from which the time-evolution and relaxation time can be obtained.

III. NEAREST NEIGHBOR KURAMOTO'S MODEL

While the above global-coupling model was successful in predicting a synchronization transition and revealed certain characteristics of the problem, in many physical situations each oscillator might only have a strong coupling with its neighbors, and the coupling is far from uniform. One might therefore naturally ask the following question: in a nearest-neighbor coupled lattice, will the synchronization transition occur, and what are the critical properties related to the transition? While nearest-neighbor coupling renders an exact solution of the system difficult, it makes the ideas of correlation length natural, and makes it possible for one to apply a renormalization group (RG) approach to the problem.

Previous work by Hong et al. [11] has examined this problem in the case of Gaussian distributed frequencies and a uniform coupling constant. For globally coupled oscillators, phase synchronization will always occur above a certain coupling threshold. However, for local coupling, there exists a possibility that the frequencies of a large proportion of the oscillators are synchronized, but the phases remain spread out across this synchronized cluster, giving rise to a frequency entrainment transition with a zero phase order parameter. Through analytical calculations in the strongly coupled linear regime and numerical simulations in the weakly coupled regime, they found that as the lattice size grows to infinity, phase synchronization occurs above a lower critical dimension of $d_l^p = 4$, while frequency entrainment has a lower critical dimension of $d_l^p = 2$. We note here that these results are reminiscent of the destruction of order in low dimensions in the Ising model. Indeed, an interesting analysis of frequency entrainment in 2D indicates that topological defects play a role in the order of the system [12], similar to the XY model.

Another possible approach to the Kuramoto model is to consider it as derived from a time-dependent Landau-Ginzburg equation [13] (a similar formulation for discrete spin systems is Glauber dynamics). This mapping will then give a Hamiltonian which has the form

$$\beta\mathcal{H} = -\sum_i \omega_i\theta_i - \sum_{<i,j>} K_{ij}\cos(\theta_j - \theta_i). \qquad (5)$$

The second term has some resemblance with the XY model Hamiltonian. However, the first term makes it difficult to transform this directly into a spin problem, so we shall not pursue this path.

IV. LOWEST CRITICAL DIMENSION: A SIMPLE RG ANALYSIS

The critical dimensions can be bounded by a simple position space renormalization group argument, first put forward by Daido [14].

Consider a generalized version of the nearest-neighbor coupled Kuramoto model, written as

$$\dot{\theta}_i = \omega_i + \sum_{j\in I_i} h_{ij}(\theta_j - \theta_i), \qquad (6)$$

where the sum here is over nearest neighbors and the coupling satisfies the symmetry $h_{ij}(-\psi) = -h_{ij}(\psi)$. Let us assume that the distribution of frequencies follows an asymptotic decay of

$$f(\omega) \propto |\omega|^{-\alpha-1} \quad (|\omega \gg 1), \qquad (7)$$

with $0 < \alpha \leq 2$. In the limit $n \to \infty$, by a generalized version of the central limit theorem [15], we shall find that

$$\hat{\omega}_n = \frac{\sum_{i=1}^N \omega_i - \gamma_n}{n^{1/\alpha}} \qquad (8)$$

will approach a stable distribution as $n \to \infty$. Since this is the main result we shall need for renormalization, $f(\omega)$ can also be chosen to have finite variance and we simply set $\alpha = 2$ for this case.

We can now perform a position space renormalization by grouping L oscillators in each direction, so that each block has $M = L^d$ oscillators. Denoting the averaged physical properties of coarse-grained oscillators with tildes on top, the coarse-grained equation is

$$\dot{\tilde{\theta}}_k = \tilde{\omega}_k + \frac{1}{M}\sum_{<i,j>l} h_{ij}(\tilde{\theta}_l - \tilde{\theta}_k + \psi_{l,i} - \psi_{k,j}), \qquad (9)$$

where $\psi_{m,j} = \theta_j - \tilde{\theta}_m$ are residual phases of individual oscillators.

Two adjacent blocks have $L^{d-1} = M^{(d-1)/d}$ adjacent oscillators, so that we can define a renormalized coupling

$$\tilde{h}_{lk}(\theta) = M^{(1-d)/d} \sum_{<i,j>} h_{ij}(\theta + \psi_{l,i} - \psi_{k,j}) \qquad (10)$$

that scales as $\mathcal{O}(1)$ with M. On the other hand, we can use Eq. 8 to write an asymptotically stable frequency. This motivates us to perform a rescaling $\tilde{t} = tM^{(1-\alpha)/\alpha}$ (this is analogous to rescaling the spin in RG for the Ising model), which shall give

$$\frac{d\tilde{\theta}_k}{d\tilde{t}} = \tilde{\omega}_{M,k} + M^{\beta} \sum_{l \in J_k} \tilde{h}_{lk}(\tilde{\theta}_l - \tilde{\theta}_k) \qquad (11)$$

where the sum runs over all blocks adjacent to k and the exponent $\beta = 1 - \alpha^{-1} - d^{-1}$.

We can carry out this procedure to $M \to \infty$, then $\tilde{\omega}_{M,k}$ describes the fixed point of the equation and for $\beta < 0$, the interaction is irrelevant, so that sufficiently large patches of oscillators become uncorrelated. We therefore find that the lower critical dimension of the frequency synchronization transition must be at least

$$d_c \geq \frac{\alpha}{\alpha - 1} \qquad (12)$$

where $1 < \alpha \leq 2$, and for $\alpha \leq 1$ no synchronization can occur. This lower critical dimension result has been verified on the Gaussian model analytically in the linear regime and numerically in general [11].

V. NUMERICAL RG IN 1D

While in 1D there will not be a full synchronization transition for an infinite size system, we can nevertheless examine the correlation length and understand its dependence on the distribution of frequencies and coupling strengths.

The advantage of examining this problem in 1 dimension is that we can perform local decimation without changing the lattice layout. This means that we can develop a local renormalization approach [16] to study synchronization and examine the universal critical properties of the system [17].

To sketch out the local decimation procedure which is analogous in spirit to operations on an antiferromagnetic chain with quenched randomness [18], let us first consider the physical picture of synchronization behaviour. Numerical simulations and intuition indicate that some oscillators will coalesce into clusters with comparable frequencies. This will happen if, relatively speaking, the coupling between pairs of oscillators is stronger than the difference in frequencies. On the other hand, if an oscillator has a large frequency, it will rapidly oscillate and the effect of neighbors will average out.

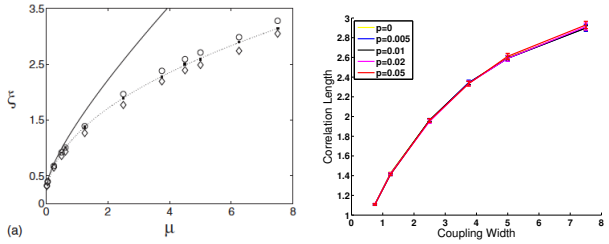

FIG. 2. (a) Comparison of simulation results and RG results for different coupling widths, with black dots from simulations, open circles from RG and diamonds from RG with strain check. The black line is a naive one-step RG estimate. Figure adapted from Ref. [16]. Our simulations found similar correlation lengths, but the RG procedure gave values that were systematically larger, which might be traced back to the slower decay in the tails than expected which would increase the correlation length. (b) Simulations for small-world networks with small p. Due to the randomness in bonds, the overall universal behavior is not modified, and the curve is only pushed up slightly.

Therefore, we can perform a fast oscillator decimation which takes out a fast oscillator, splitting the chain into two, and a strong-coupling decimation, which combines two strongly-coupled oscillators into a single one with a new effective intrinsic frequency

$$\Omega = \frac{m_n\omega_n + m_{n+1}\omega_{n+1}}{m_n + m_{n+1}}, \qquad (13)$$

where the masses m_n correspond to the number of oscillators already grouped into this single site. Due to coupling to other sites, there will be higher order corrections to the couplings and frequencies, but for the generic synchronization properties and distributions most salient features can be obtained even when ignoring these higher order terms.

The criteria for "fast" or "strong-coupling" can be estimated by considering only 2 oscillators coupled together. In this scenario, we can have a steady state with a common frequency as long as

$$|\omega_{n+1} - \omega_n| \leq \frac{K_n(m_n + m_{n+1})}{m_n m_{n+1}} = \frac{K_n}{\mu_{n,n+1}}, \qquad (14)$$

where $\mu_{n,n+1}$ is the reduced mass. Therefore, an oscillator can be considered to be fast whenever Eq. (14) is not satisfied, while otherwise the bond can be considered strong. This complementary nature makes it possible for the RG procedure to work for weak randomness as well (finite variance distributions such as Gaussians or rectangular distributions [17]) and we can perform decimation until all oscillators are gone, even though this procedure causes the coupling width to become narrower and makes the approximation less robust. When embedded in an entire chain, we shall occasionally find cases where influences from neighbors end up making this approximation incorrect, but this will be averaged out in overall statistics. When cluster sizes become large, the accumulated

phase strain along the cluster could potential split it into two, and taking this into account moderately improves the agreement with numerical data [16].

Numerical simulations have found good agreement with RG in terms of frequency distributions for different cluster sizes and the relative number of clusters of different sizes. Since the 1D system displays no frequency synchronization, it is natural to expect the cluster size distribution to obey an exponential decay. This is indeed what has been seen from numerical simulations, see Fig. 2. The simulations we performed were implemented in python using a standard fourth-order Runge-Kutta algorithm with a time step $dt = 0.01$ and a total time of $t = 250$, with the initial transient discarded. A system of $N = 10000$ oscillators was simulated for 10 distributions randomly drawn from Lorentzian frequency and coupling distributions.

Since the only control parameter here is the width of coupling (width of frequency is set to unity by a rescaling of time), one can perform a fit of the characteristic cluster decay length ξ with respect to the coupling width μ, and both simulations and RG indicate a critical scaling $\xi \sim \mu^{\nu}$ with $\nu \approx 0.5$.

For distributions with weaker randomness or finite variances, an agreement between the critical scaling exponent ν is also found between RG and simulations. By considering the flow of cluster distribution and coupling strength under the RG decimation procedure, the flow diagram and subsequently critical exponents can be analytically obtained. It has been found that the frequency and nonzero bond distributions universally approach a rectangular one and a triangular one, providing universal exponents for distributions with comparable widths [17].

VI. RG ANALYSIS OF SMALL-WORLD NETWORKS

While regular lattices provide a paradigmatic model to study nearest-neighbor effects and their scaling with dimensionality, many real world systems display a network-like structure. Random networks will have much better connectivities compared to regular lattices, and the average distance between vertices scales as the logarithm of the system size as opposed to power laws for regular lattices. Intuitively, this better connectivity will give rise to an infinite-dimension like behavior which allows a synchronization transition to occur [19].

An intermediate model between regular lattices and random networks, the small-world network, is obtained by randomly rewiring a portion p of the bonds in the network [20]. Here, we consider a Newman-Watts type small-world network [21], in which the initial lattice has k nearest neighbors connected and random bonds are added instead of rewired (when p is small, the difference between the two models is negligible). Since we can rescale the system by a finite factor k to obtain a small-world network with $k = 1$ [21] (see Fig. 3), we shall assume in the following that $k = 1$. Kuramoto's model on this type of network has only be studied with homogeneous coupling in the literature [19] and [22] with the phase synchronization and frequency synchronization transitions found to exist but with different scaling exponents with p. Here we attempt to apply some scaling arguments to understand these systems in the presence of random coupling.

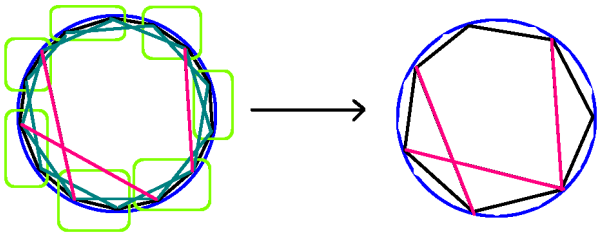

FIG. 3. Schematic of the renormalization process on a small-world network. The initial network has $k = 2$, and the nearest-neighbor, second nearest-neighbor and random bonds are colored in black, green and red. After renormalization, we can clearly see that the connection probability is increased by the rescaling factor while $k \to 1$.

We first consider a direct generalization of Daido's RG argument to small-world networks. This would entail an additional sum over the randomly added bonds, and under renormalization, we would expect to have pM as the bond probability but with K/M as the interaction strength. This procedure can continue until we reach the point where bonds are no longer scarce, and we start to see a network-like behavior, so systems of the same product pK should have comparable behavior if the strength of other lattice bonds are held the same. Unfortunately, due to time limitations we were unable to verify this numerically.

For simulations performed on a small-world network with base dimension one, Gaussian frequency distribution and with uniform coupling [22], it has been found that for finite values of p there exists a critical coupling even as the system size approaches infinity. The authors also found a scaling for phase synchronization and frequency synchronization

$$K_c^p \sim p^{-1}, K_c^f \sim p^{-1/2}, \tag{15}$$

where K_c^p is the critical coupling for phase synchronization and K_c^f is the critical coupling for frequency synchronization, which is the first Kuramoto system that can simultaneously exhibit the two different synchronization types but with different critical exponents. In the same paper, the authors also proposed a simple physical picture for phase synchronization by viewing the small-world network as regular clusters connected in a random network like structure. However, the synchronization transition explanation was more complicated and relied on a finite variance of the distribution, while no simple explanation of the scaling with N was presented.

Here, we propose a simple scaling argument to understand the behavior of the critical coupling for different values in the parameter space p, N, addressing some of the problems above. For a given system size N, a bond-adding probability p and a coupling strength of K, let us denote the proportion of the largest frequency synchronized cluster relative to the overall size of the system as $m(p, K, N)$. Note that here we mainly consider frequency synchronization, both because the aforementioned method gives a good physical picture for phase synchronization and because in section V we have already seen that the strong-coupling decimation intuition works reasonably well for cluster of frequency synchronization.

Now we consider what happens when we increase the system size to $2N$, which we view as two clusters $\{p, K, N\}$ with random bonds connecting them. There will be two cases: one in which the old cluster is roughly speaking still the largest, while the other one is where two large clusters on each side are connected by a bond to form a larger cluster. Since there are $2pN$ bonds in total with half of them between the two smaller clusters, if we define a factor q that takes into account the probability that an interconnecting bond will synchronize the two clusters on each side as well as the degeneracy factor of these largest clusters appearing on each side, we can approximately write

$$m(p, K, 2N) = \begin{cases} m(p, K, N), & \text{probability } qpNm^2(p, N) \\ \frac{m(p,K,N)}{2}, & \text{probability } 1 - qpNm^2(p, N). \end{cases} \tag{16}$$

Here we have made the assumption that $m(p, N)$ is small, since we are interested in the onset of synchronization. The factor $m(p, N)^2$ comes from requiring the two largest clusters to be connected. From this, we can find the expectation value of the new system

$$\langle m(p, K, 2N)\rangle = \frac{m(p, K, N)}{2} + \frac{q(K)pNm^3(p, K, N)}{2} \tag{17}$$

For rescaling of a factor of 2 discussed here, $m(p, K, N)/2 \leq m(p, K, 2N) \leq m(p, K, N)$, thus we posit a power law dependence $m(p, K, N) = c(p, K)N^\alpha$ at large N, where $-1 \leq \alpha \leq 0$. At large N, this shall then give us

$$c(2N)^\alpha = \frac{cN^\alpha}{2} + \frac{q(K)pc^3N^{3\alpha+1}}{2}, \tag{18}$$

or cancelling terms,

$$2^\alpha = \frac{1}{2} + \frac{q(K)pc^2(p, K)N^{2\alpha+1}}{2}. \tag{19}$$

The unstable fixed point to this equation as $N \to \infty$ is $\alpha = -1/2$. For an initial condition for this equation such that the solution for α is $\alpha < -1/2$, under subsequent steps the latter term in Eq. (19) will scale to zero and we have $\alpha = -1$. On the other hand, for an initial condition for this equation such that $\alpha > -1/2$, subsequently the last term will grow and our approximation of small $m(p, K, N)$ breaks down, giving us an asymptotic constant value of $m(p, K, N)$ as $N \to \infty$, i.e. $\alpha = 0$. We thus obtain the RG flow diagram shown in Fig. 4.

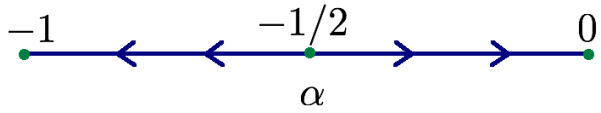

FIG. 4. RG flow for the parameter α under renormalization of system size. Depending on the inital condition, it approaches either $\alpha = 0$ or $\alpha = 1$.

This shows that with an appropriate initial condition, for any finite value of p, we can make K sufficiently large such that c is large enough for our starting scaling regime and when $N \to \infty$, we shall always find phase synchronization above this threshold. On the other hand, when $p = 0$, the second term vanishes in Eq. (19), and we must have $\alpha = -1$, which means that there is no synchronization in the 1D nearest neighbor Kuramoto model. This implies that the critical probability is $p = 0$, in accordance with the numerical results that were stated but unproved in Ref. [19, 22].

We can also try to use this formalism to obtain the critical exponents for the frequency synchronization critical coupling. As our discussion above suggests, in Eq. (19), at our starting scale N_0, the RG flow is dictated by the solution of α when solving this equation. The condition for RG flow to $\alpha = 0$ is that the initial $\alpha > -1/2$, or rewriting in terms of $m(p, K, N_0)$,

$$\frac{1}{2} + \frac{q(K)pN_0m(p, K, N_0)^2}{2} > \frac{1}{\sqrt{2}}. \tag{20}$$

To obtain the critical exponent for p, suppose the critical coupling (in the limit of large N) for frequency synchronization is $K_c^f(p)$. Let us perform a rescaling $p \to bp$; for a sufficiently large system, we can also rescale $N_0 \to N_0/b$. The critical point must thus have

$$q(K_c^f(p))m(p, K_c^f(p), N_0) = q(K_c^f(bp))m(bp, K_c^f(bp), \frac{N_0}{b}). \tag{21}$$

Potentially, one might be able to obtain the critical scaling by using a first-order approximation to these values near the critical point and plugging in the actual distribution of frequencies, which could give probabilities of bonds synchronizing two clusters. However, we were unable to obtain any quantitative information successfully. The only thing that can be perhaps hinted at is that in the generalization of Daido's RG mentioned above, the system was self-similar under a reduction of K_{random} while $K_{lattice}$ was held the same. Since the coupling strength is uniform in our problem, we would imagine that the scaling exponent would lie between the

two, which is indeed true for $\nu = -1/2$ that has been found numerically.

Finally, we would like to note that in the presence of strong randomness in the bonds as well, the critical values are drastically pushed up and one examines the correlation length. We find that for small-world networks with fewer additional bonds, the effects are indeed limited (Fig. 2(b)). Larger bond-adding probabilities were not explored due to time constraints, but could reveal more interesting dependencies.

VII. CONCLUSION AND OUTLOOK

We have examined Kuramoto's model in a nearest-neighbor coupled system with randomly distributed frequencies and coupling strengths. Nearest neighbor coupling on a lattice changes the universality of the system, leading to a lower critical dimension and the separation of phase synchronization and frequency synchronization. A position space renormalization group approach has been introduced to study frequency synchronization and shed light on the lower critical dimension, while a local decimation procedure in 1D can accurately reproduce the distribution of clusters in a system. We then use a simple RG argument to gain a better understanding of the synchronization properties of Kuramoto's model in a small world network, explaining the behaviors under scaling of different parameters of the system.

There are many more interesting explorations that can be performed. One interesting feature revealed by our analysis is that the frequency synchronization behavior on small-world networks could depend on the initial frequency distribution of oscillator strengths, which needs to be numerically verified. Another question is how coupling of mixed signs in nearest neighbor models would affect the result. In the mean-field approximation, a π-state in which the population splits in two with a π phase separation and an interesting travelling wave state in which the phases are constantly shifting can occur for mixed oscillators whose strength depends on i in Eq. (1) [23], while when the strength only depends on j in Eq. (1) the behavior is identical to Kuramoto's model [24]. It would be interesting to see whether similar behavior can be observed in nearest-neighbor lattices or networks that display phase synchronization. Kuramoto's model, the paradigm for synchronization phenomena, certainly holds much more interesting behavior for us to uncover.

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