Effect of Quantum Friction on a Particle in Periodic Potentials

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In this paper, we focus on the behavior of a particle in dissipative environment coupled to a periodic potential. We review the duality relations between the weak and strong coupling regimes, as well as perturbative RG analysis of the system, to elucidate the nature of a phase transition from a diffusive regime to a localized phase achieved by the tuning of the phenomonalogical friction coefficient. Finally, in this work, we extend the quantum friction model for a particle in a 1D periodic potential to a particle in a 2D periodic potential. We perform perturbative RG up to second order to derive the differential flow equations and phase diagrams for a particle in rectangular and rhombic lattice potentials, and observe dimensionally different confinement phases dependent on the lattice parameters of the potential.

I. INTRODUCTION

Quantum dissipation was first studied by Caldeira and Leggett to elucidate the nature of quantum tunneling in macroscopic systems, where coupling to the environment is non-negligible and strong enough to damp even classical motion[1]. Various classes of dissipative systems have become the focus of research henceforth. This includes two-level systems[2,3], which are becoming increasingly more relevant in the context of quantum computation. Extensions of basic models can now be used when analyzing the dephasing in a qubit and general decoherence in mesoscopic systems[4] . Another is the system of a particle in a one-dimensional (1D) periodic potential, relevant to the study of Luttinger liquids[5]. Studies of both system types have revealed dissipation to decrease the tunneling rate, as well as induce a transition between a diffusive regime and a delocalized phase[2,6,7].

In this paper, we focus on the latter system type. We start by using a simple picture to motivate a linearly coupled friction term. Next, we review the duality mapping between the regimes of weak and strong potential, which unveils a potential critical point for a transition of the system from a delocalized phase to a confined phase[6]. We elucidate the nature of this phase transition by reviewing the renormalization group (RG) analysis as performed previously[8]. Finally, we consider a particle in a dissipative environment coupled to a two-dimensional (2D) potential. We first consider a decoupled rectangular potential and then a coupled rhombic potential, performing perturbative RG to the second order to obtain the corresponding phase diagrams.

II. QUANTUM FRICTION THROUGH A LINEAR COUPLING TERM

In this section, we very briefly illustrate the physical origin of a linearly coupled dissipative term. Details of the calculations will be skipped over, but the reader may find them in Ref.[9]. Additionally, in this paper, we will continue to refer to the action integrated over imaginary time $\frac{1}{\hbar}S$ as βH, which is more familiar to the eyes of the readers, and furthermore drop the β prefactor in $\beta H \to H$ to minimize notational clutter.

A. Physical motivation: averaging over a fluctuating string

Let us suppose a system which consists of a particle, whose position on the x-axis $x(\tau)$ as a function of time, and a string, which extends along the y direction and whose transverse fluctuations in time are indicated by $u(\tau, y)$ (see figure 1). The particle is coupled to the end of the fluctuating string at $y = 0$ as shown. We are mainly interested in the behavior of the particle and thus trace over all fluctuations of the string to obtain an effective Hamiltonian, where effects of the string have been averaged out and the particle's properties remain the only degrees of freedom.

The Hamiltonian of the string and the particle, sepa-

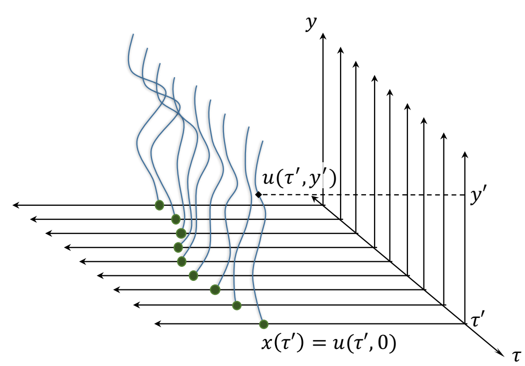

FIG. 1. Schematic of a particle coupled to one end of a fluctuating string. The effects of the motion of the string on the particle are averaged over to obtain an effective Hamiltonian whose only degrees of freedom are those of the particle.

rately, are

$$H_s = \int d\tau \int dy \left[\frac{\rho}{2}\dot{u}^2 + \frac{\sigma}{2}\Big(\frac{\partial u}{\partial y}\Big)^2\right]$$
$$H_p = \int d\tau \left[\frac{m}{2}\dot{x}^2 + U[x(\tau)]\right],$$

where ρ and σ denote the density and tension of the string, respectively, and $U[x]$ indicates some potential that the particle experiences depending on its position x. The coupling condition can be imposed on the overall partition function through the δ-function,

$$Z = \int Du(\tau, y) \int Dx(\tau)\delta\Big(x(\tau) - u(\tau, 0)\Big)e^{-H_s - H_p}.$$

Writing the δ-function as follows,

$$\delta(x(\tau) - u(\tau, 0)) = \int Df(\tau)e^{i\int d\tau f(\tau)(x(\tau) - u(\tau,0))},$$

transforming to Fourier space, and performing the Gaussian integral over $u(\omega, k)$ and subsequently over $f(\omega)$, we obtain an effective Hamiltonian for the particle such that,

$$Z = \int Dx(\omega)e^{-H_{eff}},$$
$$H_{eff} = H_p + \frac{\eta}{2}\int \frac{d\omega}{2\pi}|\omega||x(\omega)|^2, \tag{1}$$

where we have identified $\eta = 2\sqrt{\rho\sigma}$. Transforming back to time space then gives us

$$H_{eff} = H_p + \frac{\eta}{2}\int d\tau d\tau' \left(\frac{x(\tau) - x(\tau')}{\tau - \tau'}\right)^2. \tag{2}$$

Thus, we see that the string effectively induces a nonlocal friction term ($\sim \dot{x}x$) into the Hamiltonian of the particle, with η as the friction coefficient. The string in this case can be generalized to an external environment with which the particle interacts. In general, we do not know the precise details of the friction mechanism in our system, and the value of the phenomenological friction coefficient η is extracted from experimental results.

B. Mathematical motivation: available RG technology from the 2D sine-Gordon model

The quantum friction model in 1D is an analogue of the 2D sine-Gordon model. Although this point will be covered in much detail in later sections, we briefly point out the correspondence here. From eq. 1, assuming that H_p contains the standard kinetic energy term $\sim \omega^2|x(\omega)|^2$, in the long-wavelength limit, the integrand of H_{eff} is dominated by the dissipative term $\sim |\omega||x(\omega)|^2$. Thus, the two-point correlation function $\langle x(\omega)x(-\omega)\rangle \sim |\omega|^{-1}$. As a result, integrals of the form $\int d\omega\langle xx\rangle$ in perturbative analyses lead to the emergence of logarithmic interaction behaviors at large distances. This is in correspondence with the 2D sine-Gordon model, where, in the long-wavelength limit, the Hamiltonian is dominated by $\sim q^2|h(q)|^2$, which, through $\int d^2q\langle hh\rangle$ in perturbative analyses, also generates a logarithmic interaction potential[10].

As such, the techniques we used for analyzing the roughening transition can be applied to the 1D quantum friction model. These connections are crucial to the calculations carried out in the rest of this paper.

III. PARTICLE IN A DISSIPATIVE ENVIRONMENT COUPLED TO A 1D PERIODIC POTENTIAL

For a particle in a periodic potential of periodicity x_0, we rewrite eq. 2 as

$$H_0 = \frac{\eta}{4\pi\hbar}\int d\tau \int d\tau' \left[\frac{x(\tau) - x(\tau')}{\tau - \tau'}\right]^2 + \frac{1}{\hbar}\int d\tau \frac{m}{2}\dot{x}^2(\tau)$$
$$H_1 = -\frac{V}{\hbar}\int d\tau \cos\left[\frac{2\pi x(\tau)}{x_0}\right].$$

First, we follow similar notation as Fisher and Zweger[8] and scale our parameters to be dimensionless,

$$\alpha = \frac{\eta x_0^2}{2\pi\hbar}$$
$$V_0 = \frac{V}{\Lambda}$$
$$\phi(\tau) = \frac{2\pi x(\tau)}{x_0},$$

where Λ is the frequency corresponding to the quantum mechanical energy $\frac{h^2}{mx_0^2}$ required to confine a particle of mass m within a lattice spacing of x_0. Transforming to Fourier space, we arrive at the scaled Hamiltonian,

$$H_0 = \frac{1}{2}\int d\omega S(\omega)|\phi(\omega)|^2, \quad S(w) = \frac{\alpha}{2\pi}|\omega| + \frac{\omega^2}{\Lambda}$$
$$H_1 = -V_0\Lambda\int d\tau \cos\big(\phi(\tau)\big). \tag{3}$$

A. Duality Mapping

In this section, following Schmid's work, we review the duality relations between the strong and weak potential coupling regimes of the system, where $V_0 \ll 1$ and $V_0 \gg 1$, respectively. The duality mapping was one of the first studies which suggested the existence of a diffusive delocalised phase and a localized confinement phase and the corresponding critical point of transition.

1. Weak coupling expansion

For $V_0 \ll 1$, we expand e^{-H_1} as a power series, and observe its weight contribution in the integrand of the partition function $Z = \int D\phi(\tau) e^{-H_0} e^{-H_1}$,

$$\begin{aligned}
&\exp\Big(V_0\Lambda \int d\tau \cos(\phi(\tau))\Big) \\
&= \sum_n \frac{(V_0\Lambda)^n}{n!} \int \prod_{j=1}^{n} \Big[d\tau_j \Big(\frac{e^{i\phi(\tau_j)} + e^{-i\phi(\tau_i)}}{2}\Big)\Big] \\
&= \sum_n \frac{(V_0\Lambda/2)^n}{n!} \int d\tau_1 \dots d\tau_n \\
&\qquad \times \sum_{\{e_j=\pm 1\}} \exp\Big[i \int d\tau \sum_{j=1}^{n} e_j \delta(\tau - \tau_j)\phi(\tau)\Big].
\end{aligned}$$

We note here that in performing the path integral, we can separate out the zero-frequency mode, that is

$$\begin{aligned}
&\int D\phi(\tau) e^{-H} \\
&\Rightarrow \int \prod_{\omega=0}^{\infty} d\phi(\omega) \exp\Big\{ -\sum_\omega \Big[\frac{S(\omega)}{2} |\phi(\omega)|^2 \\
&\qquad - i \sum_{j=1}^{n} e_j \int d\tau \delta(\tau - \tau_j) e^{-i\omega\tau} \phi(\omega)\Big]\Big\} \\
&= \int d\phi_0 \exp\Big[i \sum_{j=1}^{n} e_j \int d\tau \delta(\tau - \tau_j)\phi_0\Big] \prod_{\omega \neq 0} d\phi(\omega) \dots,
\end{aligned}$$

where we have denoted $\phi_0 = \phi(\omega = 0)$ and used the fact that $S(0) = 0$. Thus we see that for the integral of the zero-frequency mode to not vanish and render the path integral unperformable, we require $\sum_{j=1}^{n} e_j = 0$. In other words, identifying each e_j as the charge of a classical particle and $\sum_{j=1}^{n} e_j$ as the total charge of a plasma of n such particles, only neutral plasma configurations are allowed. Thus, $n = 2N$, and we can identify a charge density

$$\rho_{2N}(\tau) = \delta(\tau - \tau_1) - \delta(\tau - \tau_2) + \delta(\tau - \tau_3) - \dots - \delta(\tau - \tau_{2N})$$

that corresponds to a neutral plasma of n particles with $(2N)!/N!N!$ different configurations originating from ways of distributing N (+) particles and N (−) particles on $2N$ τ_j sites. Thus we obtain the expression

$$\begin{aligned}
e^{-H_1} = &\sum_N \Big(\frac{(V_0\Lambda/2)^N}{N!}\Big)^2 \int d\tau_1 \dots d\tau_{2N} \\
&\qquad \times \sum_{\{e_j=\pm 1\}} \exp\Big[i \int d\tau \rho_{2N}(\tau)\phi(\tau)\Big],
\end{aligned}$$

which identifies $\rho(\tau)$ as a source term and puts the integrand of the partition function Z into Gaussian form in $\phi(\tau)$. Integrating over the fluctuations in $\phi(\tau)$, we arrive at the following,

$$\begin{aligned}
Z = \sqrt{S} &\sum_N \Big(\frac{(V_0\Lambda/2)^N}{N!}\Big)^2 \int d\tau_1 \dots d\tau_{2N} \\
&\times \exp\Big[-\frac{1}{2} \int d\tau d\tau' \rho_{2N}(\tau) S^{-1}(\tau - \tau') \rho_{2N}(\tau')\Big],
\end{aligned} \tag{4}$$

where $S = \prod_{\omega \neq 0} S(\omega)$ and, as calculated by Schmid,

$$\begin{aligned}
S^{-1}(\tau) &= \int \frac{d\omega}{2\pi} S^{-1}(\omega) e^{-i\omega(\tau)} \\
&= \begin{cases} -\frac{\Lambda}{2}|\tau|, & \alpha\Lambda|\tau| \ll 1 \\ -\frac{2}{\alpha} \ln\Big(\frac{\alpha\Lambda|\tau|}{2\pi}\Big), & \alpha\Lambda|\tau| \gg 1, \end{cases}
\end{aligned} \tag{5}$$

where we note the logarithmic behavior of the interaction at long distances.

2. Strong coupling expansion

In the limit of $V_0 \gg 1$, the tunneling of the quantum particle between adjacent minima of the periodic potential can be well described, using the WKB approximation, by its classical undamped path in the inverted potential. Denoting $H' = H - \int \frac{\omega}{2\pi} \frac{\alpha}{4\pi} |\omega| |\phi(\omega)|^2$ as the full undamped Hamiltonian, we solve the following corresponding differential equation,

$$\ddot{\bar{\phi}}(\tau) - \Lambda^2 V_0 \sin(\bar{\phi}(\tau)) = 0.$$

We arrive at the single instanton/anti-instanton solution described by

$$\bar{\phi}(\tau) = f(\tau) = \pm 4 \arctan(e^{\omega_0 \tau}),$$

where $\omega_0 = \sqrt{\Lambda^2 V_0}$. Plugging $f(\tau)$ back into H', we calculate the energy of a single (anti-)instanton $s = 8V_0^{1/2}$ which should be $\gg 1$ in the strong coupling regime. A general solution is then a combination of instantons and anti-instantons distributed at different points in time,

$$\bar{\phi}(\tau)_n = \sum_{j=1}^{n} e_j f(\tau - \tau_j),$$

where $e_j = 1$ or $e_j = -1$ describes an instanton or anti-instanton, as long as the time separation between any two instantons is substantially larger than the width of a single instanton ($|\tau_j - \tau_k| \gg w_0^{-1}$). We also note the following property of the Fourier transform of $\phi(\tau)$,

$$-i\omega\phi(\omega) = h(\omega) \sum_j e_j e^{i\omega\tau_j}, \tag{6}$$

where $h(\omega)$ is the Fourier transform of $\dot{f}(\tau)$. Then, substituting ϕ_n into the full Hamiltonian H, we obtain the following expression

$$H[\phi_n] = ns + \frac{1}{2}\sum_{jk} e_j e_k \Delta(\tau_j - \tau_k),$$

where the second term describes an effective pairwise interaction between the instantons and anti-instantons, and

$$\begin{aligned}\Delta(\tau) &= \frac{\alpha}{2\pi}\int \frac{d\omega}{2\pi}\frac{|h(\omega)|^2}{|\omega|}e^{-i\omega(\tau)} \\ &= \begin{cases} -c\frac{\alpha}{2\pi}\omega_0^2\tau^2, & \omega_0|\tau| \ll 1 \\ -2\alpha\ln(\omega_0\tau), & \omega_0|\tau| \gg 1, \end{cases}\end{aligned} \tag{7}$$

where c is a constant of order unity. First, we note that since $h(\omega = 0) = 2\pi$ is a constant, integration over the zero-frequency mode leads to a divergence in energy unless $\sum_j e_j = 0$. Therefore, only solutions with equal numbers of instantons and anti-instantons are allowed, imposing the condition $n = 2N$ as with the weak coupling expansion.

Next, taking into account Gaussian fluctuations around $\bar{\phi}$, the work for which is not shown here but can be found in various reference texts[9], the fugacity e^{-s} is replaced with $\omega_0(2s/\pi)^{1/2}e^{-s}$. Thus, identifying $\rho_{2N}(\tau)$ again as the density of a "neutral" mixture of instantons and anti-instantons, we arrive at the following expression for the partition function,

$$\begin{aligned} Z = \sum_N &\left(\frac{(w_0 e^{-s}\sqrt{2s/\pi})^N}{N!}\right)^2 \int d\tau_1 \ldots d\tau_{2N} \\ &\times \exp\left[-\frac{1}{2}\int d\tau d\tau' \rho_{2N}(\tau)\Delta(\tau-\tau')\rho_{2N}(\tau')\right]. \end{aligned} \tag{8}$$

We first compare the effective interaction potentials of the corresponding charge and instanton densities $\rho_{2N}(\tau)$ in equations 5 and 7. We see that just as in the weak coupling expansion, a logarithmic interaction is generated between the instantons and anti-instantons separated by large distances in time in the strong coupling expansion. If we neglect their behavior in the core regions (for small $|\tau|$), we are able to map them onto each other through the relation,

$$\alpha \to \frac{1}{\alpha}. \tag{9}$$

Next, comparing the partition functions in 4 and 8, we also recognize the mapping of the prefactor raised to the nth power in the power sums,

$$\frac{8}{\sqrt{\pi}}e^{-s}V_0\Lambda \to V_0\Lambda. \tag{10}$$

Mappings 9 and 10 then lead to the following fixed points, as shown in the top of figure 2,

$$\begin{aligned} \alpha^* &= 1 \\ V_0^* &= \left(\frac{\ln(8/\sqrt{\pi})}{8}\right)^2 \approx 0.04. \end{aligned} \tag{11}$$

However, we point out that while the weak coupling expansion did not assume conditions of the system parameters, the expansion from strong coupling assumed the use of the WKB approximation to be valid. Here, as $V_0^* \approx 0.04$, our fixed point is not in the strongly coupled regime where the latter condition is satisfied. Therefore, the estimate of α^* is also not completely reliable, motivates further analysis using RG.

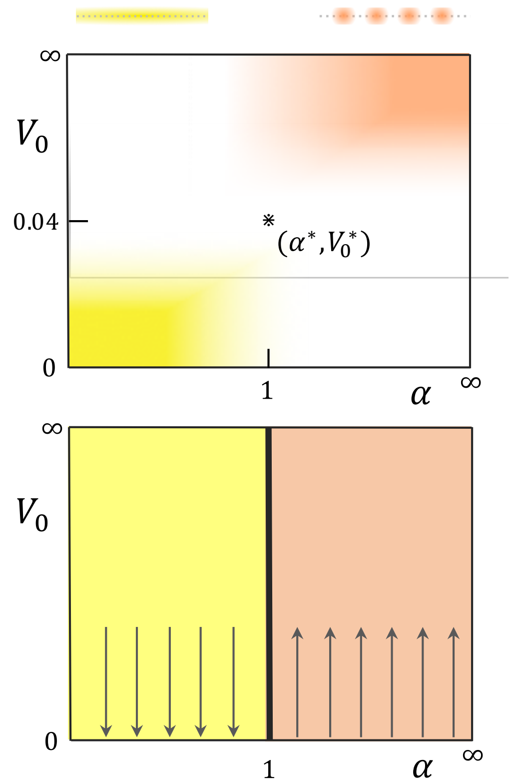

FIG. 2. Schmid identified a fixed point at $\alpha = 1$ by duality mapping between the strong and week potential coupling regimes (top). The nature of the phase transition between a delocalized and confined phase was clarified through RG analysis, such as in the work of Fisher and Zweger (bottom).

B. Perturbative RG analysis under a 1D periodic potential

Before performing RG, we point out the, crucially, the periodic potential imposes two constraints on our system. First, that rescaling in the time coordinate during RG should not affect the real space coordinate on which our particle rests. The periodic nature of our system is constrained by our system size in real space, which does not change under renormalization in time. Thus, x_0 stays the same under renormalization and is not treated as a scalable parameter but rather a given system constant. Secondly, a sensible renormalization procedure will require the dimensionless wave function $\phi(\tau)$ to map back onto itself with the same periodicity $\phi \to \phi + 2\pi$, else breaking the symmetry of the system. To preserve this symmetry, the renormalization constant of $\phi(\tau)' = \phi(\tau)/z$ is $z = 1$.

An important consequence follows. Since $\phi(\tau)$ does not get renormalized, by power counting, we can see that under rescaling $\omega \to \omega b^{-1}$, the quadratic term $\sim \omega^2$, as well as any terms that are higher order in ω, is irrelevant in the presence of the friction term in the 1D bare Hamiltonian. Thus, in our following calculations and analysis, we take the limit $S(\omega) \to \alpha|\omega|/2\pi$. The two-point correlation function in frequency space is then,

$$\langle \phi(\omega)\phi(\omega')\rangle = \frac{2\pi\delta(\omega+\omega')}{S(\omega)}. \tag{12}$$

We perform the standard renormalization procedure by separating the fast and slow modes of $\phi(\omega)$,

$$\phi(\omega) = \begin{cases} \phi_<(\omega), & \text{for } |\omega| < \Lambda/b \\ \phi_>(\omega), & \text{for } \Lambda/b \le |\omega| < \Lambda, \end{cases}$$

allowing us to identify these modes in time space as

$$\begin{aligned} \phi(\tau) &= \int_0^{\Lambda/b} \frac{d\omega}{2\pi}\phi(\omega)e^{i\omega\tau} + \int_{\Lambda/b}^{\Lambda} \frac{d\omega}{2\pi}\phi(\omega)e^{i\omega\tau} \\ &= \phi_<(\tau) + \phi_>(\tau) \end{aligned}$$

The two-point correlation function of the fast modes is then

$$G(\tau) \equiv \langle \phi_>(\tau)\phi_>(0)\rangle = \int_{\Lambda/b}^{\Lambda} \frac{d\omega}{2\pi}\frac{e^{i\omega\tau}}{S(\omega)}, \tag{13}$$

which we will use to integrate out the corresponding fast fluctuations. In obtaining the $G(\tau)$, Fisher and Zweger incorporates a smoothing function $W(\omega)$ into the integral above, where $W(\omega) = 1$ when $\omega \gg \Lambda/b$ and vanishes slowly as $\omega \to \Lambda/b$, and is used to prevent the generation of spurious long-range behavior in $G(\tau)$. Here, we will take the results as calculated by Fisher and Zweger for use in later calcuations:

$$\begin{aligned} G(\tau) &= \frac{2}{\alpha}K_0(\Lambda\tau/b) \quad \text{for } \Lambda\tau \gg 1, \\ G(0) &= \frac{2}{\alpha}\ln(b), \end{aligned}$$

where $K_0(z)$ is the modified Bessel function of the second kind, which decays exponentially for large z and is even in z.

From here, we study the effects of the perturbation terms in

$$H = H_0 + \langle H_1\rangle_> - \frac{1}{2}\Big(\langle H_1^2\rangle_> - \langle H_1\rangle_>^2\Big) + O(V_0^3). \tag{14}$$

Using $G(\tau)$, we calculate the first order perturbation term $\langle H_1\rangle$ and rescale $\tau' = \tau b$.

$$\begin{aligned} \langle H_1\rangle &= -V_0\Lambda \int d\tau \langle \cos\big(\phi_<(\tau)+\phi_>(\tau)\big)\rangle \\ &= -V_0\Lambda \int d\tau \Big[\cos\phi_<\langle\cos\phi_>\rangle - \sin\phi_<\langle\sin\phi_>\rangle\Big] \\ &= -V_0\Lambda \int d\tau \cos\phi_< \left[\frac{\langle e^{i\phi_>}\rangle + \langle e^{-i\phi_>}\rangle}{2}\right] \\ &= -V_0\Lambda \int d\tau \cos\phi_< \Big[e^{-\frac{1}{2}G(0)}\Big] \\ &= -V_0\Lambda b^{1-\frac{1}{\alpha}} \int d\tau' \cos(\phi_<(\tau)), \end{aligned}$$

where in line 2, $\langle\sin\phi_>\rangle$ vanishes since $\sin\phi$ is odd in ϕ. Therefore, first order perturbation rescales V_0 as $V_0(b) = b^{1-\frac{1}{\alpha}}V_0$ and does not rescale α. Upon linearizing $b \approx 1+\delta l$, we arrive at the following differential flow equation,

$$\frac{\partial V_0(l)}{\partial l} = \Big(1 - \frac{1}{\alpha}\Big)V_0(l), \tag{15}$$

and observe that for V_0 is irrelevant and scales to 0 if $\alpha < 1$ and grows if $\alpha > 1$.

Next, we calculate the second order perturbation term. Denoting $\phi(\tau)$ and $\phi(\tau')$ as ϕ and ϕ', respectively,

$$\begin{aligned} \langle H_1^2\rangle &= (V_0\Lambda)^2 \int d\tau d\tau' \langle \cos\big(\phi_< + \phi_>\big)\cos\big(\phi'_< + \phi'_>\big)\rangle \\ &= (V_0\Lambda)^2 \int d\tau d\tau' \Big[\cos\phi_<\cos\phi'_<\langle\cos\phi_>\cos\phi'_>\rangle \\ &\qquad + \sin\phi_<\sin\phi'_<\langle\sin\phi_>\sin\phi'_>\rangle\Big] \\ &= (V_0\Lambda)^2 \int d\tau d\tau' \frac{1}{2}\Big[\cos(\phi_< - \phi'_<)\langle\cos(\phi_> - \phi'_>)\rangle \\ &\qquad + \cos(\phi_< + \phi'_<)\langle\cos(\phi_> + \phi'_>)\rangle\Big] \\ &= (V_0\Lambda)^2 \int d\tau d\tau' \frac{e^{-G(0)}}{2}\Big[\cos(\phi_< - \phi'_<)e^{G(\tau-\tau')} \\ &\qquad + \cos(\phi_< + \phi'_<)e^{-G(\tau-\tau')}\Big], \end{aligned}$$

and thereby,

$$\begin{aligned} \langle H_1^2\rangle - \langle H_1\rangle^2 = &\frac{(V_0\Lambda)^2}{2}e^{-G(0)} \int d\tau d\tau' \\ \times\Big[&\cos(\phi_< - \phi'_<)(e^{G(\tau-\tau')} - 1) + \cos(\phi_< + \phi'_<)(e^{-G(\tau-\tau')} - 1)\Big]. \end{aligned} \tag{16}$$

By gradient expansion, the first term $\sim \left(\frac{\partial\phi}{\partial\tau}\right)^2 \sim \omega^2$, and is therefore irrelevant by power counting, as with the quadratic term in H_0. The second term $\sim \cos(2\phi)$. This is the analogue of the symmetry-breaking term $h_p\cos(p\theta(x))$ in the 2D Sine Gordon model. Thus, for any term with the form $V_n\cos(n\phi)$ in our model, we make the following correspondence to the relation derived in our sixth problem set,

$$\frac{dh_p}{dl} = \left(2 - \frac{p^2}{4\pi K}\right)h_p \quad \Longleftrightarrow \quad \frac{dV_i}{dl} = \left(1 - \frac{n^2}{\alpha}\right)V_i, \tag{17}$$

where $2 \to 1$ in the first term comes from going from 2 dimensional space to 1 dimensional time, and $4\pi K \to \alpha$ is found by comparing the two-point correlation functions of the bare Hamiltonian of both models.

Thus, for the $V_n \cos n\phi$ term to be relevant, we need $\alpha > n^2$. However, for the perturbation relations in V_0 to be valid, we need $\alpha < 1$, as shown by the first order relation in equation 15. Therefore, $\alpha > n^2$ cannot be satisfied and the $V_n \cos(n\phi)$ term, $\sim \cos(2\phi)$ in this specific case, is not relevant. Therefore, to second order, neither V_0 nor α is rescaled.

In fact, as argued by Fisher and Zweger, we do not expect α to be rescaled to any order in V_0. As defined, α depends only on η and x_0. As we have pointed out before, x_0 is a system constant that does not change under renormalization. On the other hand, η is the coefficient of $|\omega||\phi(\omega)|^2$, a term that cannot be generated by derivatives of $\phi(\tau)$ and is thus nonlocal, as explicitly apparent in its Fourier transform: $\left(\frac{x(\tau)-x(\tau')}{\tau-\tau'}\right)^2$. Therefore, since the perturbative RG procedure is only capable of generating local interactions, η and thus α remains unaltered to all orders in V_0. This is markedly different from the behavior of the 2D Sine-Gordon model. Even though both use the same technology regarding the 2D coulomb gas renormalization group, the $q^2|h(q)|^2$ term in the 2D Sine-Gordon model is local and thus the temperature coefficient $K \sim T^{-1}$ is rescaled under renormalization.

In the flow diagram, this means the line of the phase transition is vertical, as shown in figure 2. The particle is spatially delocalized in the regime of low friction ($\alpha < 1$), and spatially confined in the regime of high friction ($\alpha > 1$).

IV. PARTICLE IN 2D PERIODIC POTENTIALS

In the following section, we will extend RG analysis of the particle in 1D periodic potential to that in a 2D periodic potential.

A particle in two dimensions has two orthogonal modes: $x(\tau)$ and $y(\tau)$. For our analysis, we assume that the friction coefficient itself, η, is isotropic. Then, given lattice parameters x_0 and y_0, we rescale our dimensionless parameters accordingly,

$$\begin{aligned} \alpha_x &= \frac{\eta x_0^2}{2\pi\hbar}, & \alpha_y &= \frac{\eta y_0^2}{2\pi\hbar}, \\ V_0^x &= \frac{V^x}{\Lambda}, & V_0^y &= \frac{V^y}{\Lambda}, \\ \phi_x(\tau) &= \frac{2\pi x(\tau)}{x_0}, & \phi_y(\tau) &= \frac{2\pi y(\tau)}{y_0}. \end{aligned}$$

A. Uncoupled modes: rectangular lattice potential

A rectangular lattice potential, with adjacent minima in x and y direction separated by x_0 and y_0, respectively, is given by

$$\begin{aligned} H_1 &= \int d\tau V_0^x \cos\left(\frac{2\pi x(\tau)}{x_0}\right) + V_0^y \cos\left(\frac{2\pi y(\tau)}{y_0}\right) \\ &= H_1^x(x) + H_1^y(y). \end{aligned} \tag{18}$$

We can completely rewrite the Hamiltonian as separate functions of $\phi_x(\tau)$ and $\phi_y(\tau)$:

$$H = H_x(\phi_x) + H_y(\phi_y), \tag{19}$$

where,

$$H_i(\phi_i) = \frac{1}{2}\int d\omega \left(\frac{\alpha_i}{2\pi}|\omega| + \frac{\omega^2}{\Lambda}\right)|\phi_i(\omega)|^2 - V_0^i \Lambda \int d\tau \cos\big(\phi_i(\tau)\big).$$

Now we can solve for each mode independently as we did for the particle in one dimensions.

Performing the same RG analysis for each mode gives us two sets of conditions regarding α_x and α_y separately, imposing conditions regarding the relevance of V_0^x and V_0^y, respectively. Since α_x and α_y are just rescaled versions of the same phenomenological parameter η, we obtain the following conditions,

$$\frac{\eta}{2\pi} \begin{cases} \begin{cases} < \frac{1}{x_0^2}, & V_0^x \text{ is irrelevant} \\ > \frac{1}{x_0^2}, & V_0^x \text{ is relevant} \end{cases} \\ \\ \begin{cases} < \frac{1}{y_0^2}, & V_0^y \text{ is irrelevant} \\ > \frac{1}{y_0^2}, & V_0^y \text{ is relevant} \end{cases} \end{cases}$$

For a square lattice potential where $x_0 = y_0$, the two sets of conditions are identical, and we obtain the same phase diagram as that of the one dimensional case, with a delocalized phase and confined phase. For rectangular lattice potentials where $x_0 \neq y_0$, we get that the relevance of V_0^x and V_0^y depend differently on η. More concretely, denoting η_i as the minimum value of η that leads to confinement of the particle in the i-direction, and letting $x_0 > y_0$, we see that $\eta_x < \eta_y$. Therefore, we observe a picture, where, along the direction where the adjacent minima are farther apart, it is harder for the particle to tunnel and thus it requires less friction to get stuck. The possible phase diagrams, in a plane with fixed coordinates, are shown in figure 3.

B. Coupled modes: rhombus lattice potential

Now we consider rhombic potentials obtained by the multiplication of periodic functions,

$$H_1 = V_0\Lambda \int d\tau \cos\Big(\phi_x(\tau)\Big)\cos\Big(\phi_y(\tau)\Big),$$

where x_0 and y_0 are again the distance between the adjacent potential minima in the x and y direction, respectively. Except in contrast to the rectangular lattice, each "layer" is staggered (see schematic in figure 4).

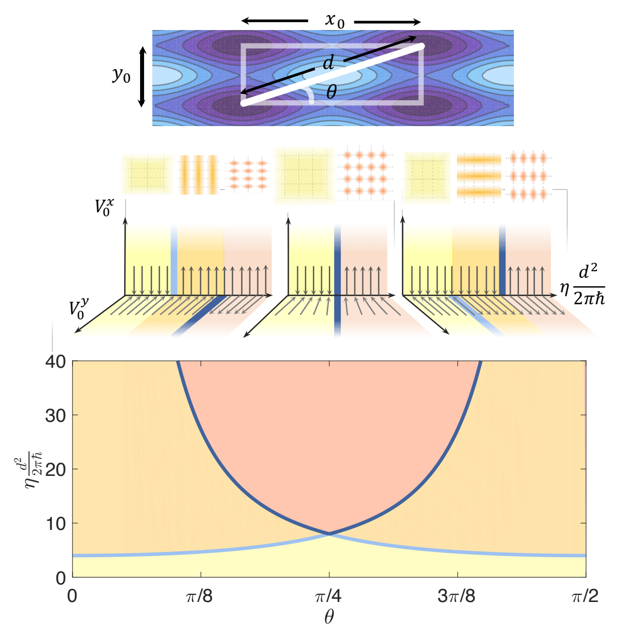

FIG. 3. The phase diagram of a particle in a dissipative environment coupled to a rectangular 2D potential. For a non-square potential, the particle experiences a phase of 1D confinement (orange) between passing into the delocalized phase (yellow) or the fully confined 2D phase (red).

We cannot perform coordinate transformation to decouple modes in H_1 without inducing mode-mixing in H_0. That is, the following transformation $\phi'_x = \phi_x + \phi_y$, $\phi'_x = \phi_x - \phi_y$ gives us decoupled H_1,

$$H_1 = \frac{V_0 \Lambda}{2} \int d\tau cos(\phi'_x(\tau)) + cos(\phi'_y(\tau)),$$

but rearranges H_0 as

$$H_0 = \int \frac{d\omega}{2\pi} \frac{1}{4\pi} \Bigg[(\alpha_x + \alpha_y) \Big(|\phi'_x(\omega)|^2 + |\phi'_y(\omega)|^2 \Big) + (\alpha_x - \alpha_y) \Big(\phi'_x(\omega)\phi'_y(-\omega) + \phi'_x(-\omega)\phi'_y(\omega) \Big) \Bigg].$$

The mode-mixing term disappears when $x_0 = y_0$, which reduces to the case of a square lattice rotated by 90^o, with inter-minima spacing of $x_0/\sqrt{2}$. We perform perturbative RG to calculate the effect of H_1 on the system parameters.

1. *Perturbative RG analysis*

First, we identify the two-point correlation functions of the bare Hamiltonian upon integrating over the fast modes $G_i(\tau) \equiv \langle \phi_i(\tau)\phi_i(0) \rangle$

$$\begin{aligned} G_i(\tau) &= -\frac{2}{\alpha_i} K_0(\Lambda\tau/b) \quad \text{for } \Lambda\tau \gg 1, \\ G_i(0) &= \frac{2}{\alpha_i} \ln(b), \end{aligned} \tag{20}$$

where $i = x$ or y and also noting $\langle \phi_x(\tau)\phi_y(0) \rangle = 0$.
Calculation is as follows,

$$\begin{aligned} \langle H_1 \rangle &= V_0 \Lambda \int d\tau \langle \cos\left(\phi_{x,<} + \phi_{x,>}\right) \cos\left(\phi_{y,<} + \phi_{y,>}\right) \rangle \\ &= V_0 \Lambda \int d\tau \frac{1}{2} \Big[\cos(\phi_{x,<} - \phi_{y,<}) \langle \cos(\phi_{x,>} - \phi_{y,>}) \rangle \\ &\qquad + \cos(\phi_{x,<} + \phi_{y,<}) \langle \cos(\phi_{x,>} + \phi_{y,>}) \rangle \Big] \\ &= V_0 \Lambda \int d\tau e^{-\frac{1}{2}(G_x(0) + G_y(0))} \Big[\cos(\phi_{x,<}) \cos(\phi_{y,<}) \Big] \end{aligned}$$

Rescaling $\tau' = \tau b$ and linearizing $b \approx 1 + \delta l$, we have that

$$\frac{\partial V_0(l)}{\delta l} = \left[1 - \left(\frac{1}{\alpha_x} + \frac{1}{\alpha_y} \right) \right] V_0(l).$$

Thus, V_0 scales to 0 when $\eta < 2\pi\hbar(\frac{1}{x_0^2} + \frac{1}{y_0^2})$. We check that when $x_0 = y = 0$, the imposed condition becomes identical to that of a square lattice with spacing $x_0/\sqrt{2}$. Physically, we expect the anisotropy of the potential to reflect in the system in some way, which motivates us to seek higher order effects. To minimize clutter in the following calculations, we will drop the $<$ label on the slow modes and transform notations for the fast modes as such $\cos(\phi_{x,>}(\tau')) \Rightarrow c'_x$.

$$\begin{aligned} \langle H_1^2 \rangle &= (V_0 \Lambda)^2 \int d\tau d\tau' \langle (\cos(\phi_x) \cos(\phi_y) \cos(\phi'_x) \cos(\phi'_y) \rangle \\ &= (V_0 \Lambda)^2 \int d\tau d\tau' \frac{1}{4} \\ &\times \Big[\cos(\phi_x - \phi_y) \cos(\phi'_x - \phi'_y) \langle (c_x c_y + s_x s_y)(c_x c_y + s_x s_y) \rangle \\ &+ \cos(\phi_x - \phi_y) \cos(\phi'_x + \phi'_y) \langle (c_x c_y + s_x s_y)(c_x c_y - s_x s_y) \rangle \\ &+ \cos(\phi_x + \phi_y) \cos(\phi'_x - \phi'_y) \langle (c_x c_y - s_x s_y)(c_x c_y + s_x s_y) \rangle \\ &+ \cos(\phi_x + \phi_y) \cos(\phi'_x + \phi'_y) \langle (c_x c_y - s_x s_y)(c_x c_y - s_x s_y) \rangle \Big]. \end{aligned}$$

The second and third terms are identical upon the exchange $\tau \leftrightarrow \tau'$. The expectation values of the first and fourth terms are identical because cross terms such as $\langle c_x s_x \rangle$ has an argument that is odd in ϕ_x and thus vanish. The expectation values in the first and fourth term are calculated to be,

$$\begin{aligned} &\langle (c_x c_y + s_x s_y)(c_x c_y + s_x s_y) \rangle = \langle (c_x c_y - s_x s_y)(c_x c_y - s_x s_y) \rangle \\ &= e^{-(G_x(0) + G_y(0))} \cosh \Big[G_x(\tau - \tau') + G_y(\tau - \tau') \Big]. \end{aligned}$$

The expectation values in the first and fourth term are calculated to be

$$\begin{aligned} &\langle (c_x c_y + s_x s_y)(c_x c_y - s_x s_y) \rangle \\ &= e^{-(G_x(0) + G_y(0))} \cosh \Big[G_x(\tau - \tau') - G_y(\tau - \tau') \Big]. \end{aligned}$$

The slow-mode function from the first and fourth terms are then collected to be

$$\cos(\phi_x-\phi_y)\cos(\phi'_x-\phi'_y)+\cos(\phi_x+\phi_y)\cos(\phi'_x+\phi'_y)$$
$$=\cos(\phi_x+\phi'_x)\cos(\phi_y+\phi'_y)+\cos(\phi_x-\phi'_x)\cos(\phi_y-\phi'_y). \quad (21)$$

The slow-mode function from the second and third terms are collected to be

$$2\cos(\phi_x-\phi_y)\cos(\phi'_x+\phi'_y)$$
$$=\cos(\phi_x+\phi'_x)\cos(\phi_y-\phi'_y)+\cos(\phi_x-\phi'_x)\cos(\phi_y+\phi'_y), \quad (22)$$

where in the collection of these terms, we have removed functions that are odd under the exchange of $\tau \leftrightarrow \tau'$. Approximating $cos(\phi_i-\phi'_i)$ by the gradient expansion and removing terms $(\frac{\partial\phi_i}{\partial\tau})^2$ by the same reasoning of power counting as before, we see that second order RG generates the following terms,

$$V_0^{(2)}\cos(2\phi_x)\cos(2\phi_y)+V_0^x\cos(2\phi_x)+V_0^y\cos(2\phi_y), \quad (23)$$

where, using results 21 and 22, the coefficients are, at this order,

$$V_0^{(2)}=\frac{(V_0\Lambda)^2}{4}e^{-(G_x(0)+G_y(0))}\times\left\{\cosh\left[G_x(\tau-\tau')+G_y(\tau-\tau')\right]-1\right\}$$
$$V_0^x=V_0^y=\frac{(V_0\Lambda)^2}{4}e^{-(G_x(0)+G_y(0))}\times\left\{\cosh\left[G_x(\tau-\tau')-G_y(\tau-\tau')\right]-1\right\}$$

Now, we analyze the relevance of each of the generated terms. We immediately see that $V_0^{(2)}\cos(2\phi_x)\cos(2\phi_y)$ is irrelevant, and comment that this is true for any term of the form $V_0^{(n)}\cos(n\phi_x)\cos(n\phi_y)$ where $n>1$. In fact, we see that, generally, a term of the form rescales $\cos(n_x\phi_x)\cos(n_y\phi_y)$ effectively rescales $x_0\to x_0/n_x$ and $y_0\to y_0/n_y$ and is relevant in the presence of our current potential term $V_0\cos(\phi_x)\cos(\phi_y)$ if the following condition is satisfied,

$$\frac{n_x^2}{\alpha_x}+\frac{n_y^2}{\alpha_y}<\frac{1}{\alpha_x}+\frac{1}{\alpha_y}. \quad (24)$$

Thus, denoting $\tan(\theta)\equiv y_0/x_0$, we arrive at the conditions that $V_0^x\cos(2\phi_x)$ $(n_x=2,n_y=0)$ is relevant if $\theta<\pi/6$, and $V_0^y\cos(2\phi_y)$ $(n_x=0,n_y=2)$ is relevant if $\theta>\pi/3$. Here, we ponder if additional, different, terms can be generated by higher order perturbation. However, we see that the only terms that have not yet been generated and which have possible lattice configurations that can satisfy condition 24 are those of $(n_x>2,n_y=0)$ and $(n_x=0,n_y>2)$. However, these terms are irrelevant in the presence of the $\cos(2\phi_i)$ terms that have already been generated. Now, denoting $d^2=x_0^2+y_0^2$, we summarize the nature of the phase transitions for different ranges of θ in the following table.

θ range	phases	transition $\frac{d^2}{2\pi\hbar}\cdot\eta(\theta)$
$0<\theta<\frac{\pi}{6}$	delocalized 1D confinement along x 2D confinement	——— $\sec^2(\theta)$ ——— —– $\sec^2(\theta)\csc^2(\theta)$—–
$\frac{\pi}{6}<\theta<\frac{\pi}{3}$	delocalized 2D confinement	—– $\sec^2(\theta)\csc^2(\theta)$—–
$\frac{\pi}{3}<\theta<\frac{\pi}{2}$	delocalized 1D confinement along y 2D confinement	——— $\csc^2(\theta)$ ——— —– $\sec^2(\theta)\csc^2(\theta)$—–

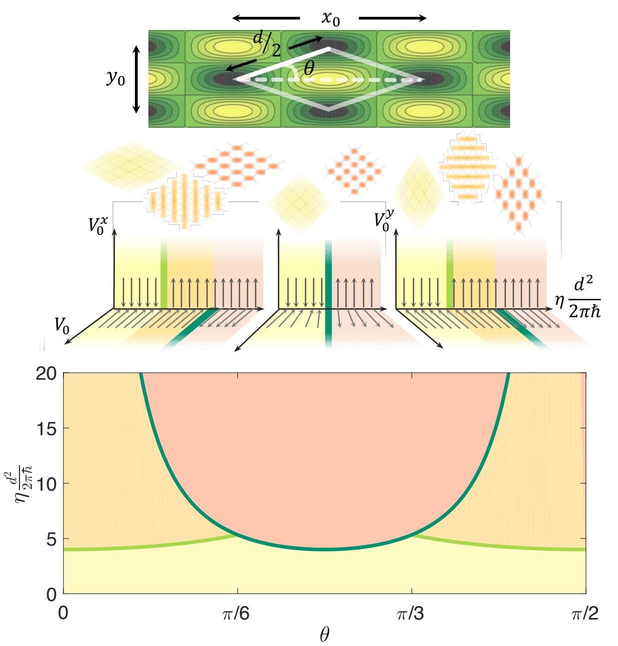

FIG. 4. The phase diagram of a particle in a dissipative environment coupled to a rhombic 2D potential. We see that there is a finite parameter space in θ, including not just the diamond configuration, where the the system transitions directly from a completely diffuse phase (yellow) to a completely confined phase (red), skipping over the partial 1D confinement phase (orange).

Thus, as shown in figure 4, for a rhombic 2D potential, there exists a finite parameter space in θ where the the system transitions directly from a completely diffuse phase (yellow) to a completely confined phase (red), skipping over the partial 1D confinement phase (orange).

Additionally, we can replace d in the scaling coefficient of η in the rectangular case to $\bar{d}=d/\sqrt{2}$, while keeping $\bar{d}=d$ in the rhombic case. In this way, the rectangular and rhombic potentials exactly map onto each other at $\theta=\pi/4$ as the square lattice potential. As shown in figure 5, they also confirm the same position on the phase diagram.

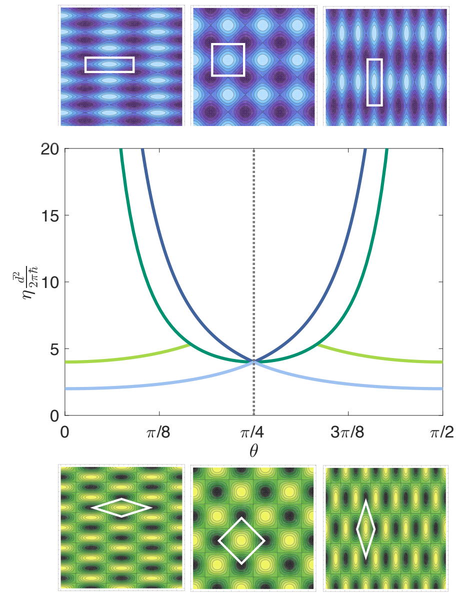

FIG. 5. A comparison of the phase diagrams for a particle in a dissipative environment coupled to a rectangular and to a rhombic potential. From the grey dotted line, we see that at $\theta = \pi/4$, our results for the rectangular and rhombic potentials map back onto each other, as expected.

V. CONCLUSION AND OUTLOOK

In conclusion, we have introduced a dissipative term into our Hamiltonian and reviewed the duality mapping and phase transitions of a quantum dissipative system coupled to a 1D periodic potential. We employed perturbative RG to examine a particle in a dissipative environment coupled to rectangular and rhombic 2D periodic potentials, and identified the corresponding phase transitions.

The analysis in section IV can be extended to 3D potentials of the corresponding form. The author is in the process of carrying out these calculations. As well, it will be interesting to see if these higher dimensional dissipative systems with periodic potentials are relevant to or accessible in any experiments.

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