Failure of the Conformal Bootstrap Technique in Fractional Dimensions Below Two

Patrick J. Ledwith*
MIT Department of Physics
(Dated: May 20, 2017)

We study a conformal bootstrap approach to the Ising model in fractional dimension $1 < d < 2$. In particular, we propose that the conformal bootstrap as it stands is not a natural approach to define and analyze conformal field theories in dimension below two. We then use these ideas to suggest a mechanism for the numerical results obtained for $1 < d < 2$ [1]. Finally, we discuss a potential epsilon expansion around the exact solution at $d = 2$, finding that a Hamiltonian approach for such a method would not be tractable and that methods using conformal perturbation theory are likely the only way forward.

The recent development of the conformal bootstrap has resulted in its rapid adoption into studies of critical behavior. Critical behavior only implies scale invariance, but most critical points, such as that of the Ising model, are also conjectured or proved to have conformal symmetry as well. This implies that, after coarse graining to obtain a field theory, a system at its critical point can be descried via a conformal field theory.

Analysis of critical behavior through conformal field theory has the theoretical advantage that it allows one to study universality classes themselves, rather than the models that belong to them, thereby focusing on relevant, universal behavior from the start. It is therefore a much cleaner approach than conventional methods. Recent advances in numerics have allowed it to become practically useful as well: it currently provides the best known estimates for the critical exponents of the three dimensional Ising model [2]. It also provides a method for nonperturbatively calculating critical exponents in fractional dimensions [3]. Unfortunately, this method seems to fail in $1 < d < 2$: Ising critical exponents are very clearly ruled out. In this paper, we propose a reason why, and begin discussing a method to further investigate the behavior in this regime.

The paper is structured as follows. In the first section, we review the formalism of the conformal bootstrap, specifically the pieces relevant to understanding the bootstrap calculations of Ising exponents. In the second section, we specialize to the case of the Ising model and explain how one can pick out the Ising model from the zoo of $\mathbb{Z}_{\not{2}}$ symmetric conformal field theories. Next, in the third section, we discuss the analytic continuation to fractional spacial dimension and subtleties that occur when $d < 2$. In the fourth section, we interpret the numerical results using our proposal that the numerical method ceases to be valid for $d < 2$ because it adds extra constraints. Finally, in the fifth section, we discuss a potential method to analytically explore behavior close to $d = 2$ and set up the very beginning of the calculation.

* pled@mit.edu

I. THE CONFORMAL BOOTSTRAP AND THE ISING MODEL

In this section, we review the key elements of conformal bootstrap method. What we present is a strict subset of [4], where readers can look to find detailed proofs of the statements given below.

Conformal symmetry for $d > 2$ consists of translation symmetry generated by P_μ, rotational symmetry generated by $M_{\mu\nu}$, dialation symmetry generated by D, and symmetry with respect to special conformal transformations generated by K_μ. The case for $d = 2$ is more complicated, and is analyzed using the Virosoro algebra.

We are interested in understanding euclidean conformal field theory and in particular the scaling of operators. This corresponds to the eigenvalues of the dialation operator, and so we seek a formalism where it is naturally diagonal.

In Euclidean signature, by rotational invariance, we can choose any direction we want to represent "time". Since we are interested in the eigenvalues of the dialation operator, it makes sense to choose the distance from the origin to represent time, so that the dialation operator generates the evolution of the states. This is known as radial quantization and the resulting states live on spheres that evolve by increasing their radii. Using conformal invariance, this construction can be used to define a one to one correspondence between an operator and the state living on the sphere containing an insertion of that operator. This construction gives $|O\rangle = O\,|0\rangle$ as the state corresponding to the operator O.

We are interested mostly in the dialation operator and so work with a basis $|O_i\rangle$ such that it is diagonal. The eigenvalues are labeled Δ_i and correspond to the scaling dimension of the operator O_i. The scaling dimensions of the relevent operators are sufficient to determine the critical exponents, and are therefore our primary concern.[1] With the above definitions, one can calculate

$$[D, M_{\mu\nu}] = 0$$

[1] The condition of relevancy in the operator language translates to $y_O = d - \Delta_O > 0$.

$$[D, P_\mu] = P_\mu \tag{1}$$

$$[D, K_\mu] = -K_\mu.$$

The first relation tells us that we can also label operators according to which representation of $SO(d)$ they belong to. The second two relations tell us that P_μ and K_μ act as raising and lowering operators respectively. We call operators O whose states are annihilated by K_μ primary, and say that the operators corresponding to states of the form $P_{\mu_1}...P_{\mu_n}|O\rangle$ as members of the conformal multiplet generated by O.

Conformal symmetry can also be used to fix the form of correlation functions. The two point correlation functions of two primary scalar operators is given by

$$\langle O_i O_j \rangle = \frac{C\delta_{\Delta_i \Delta_j}}{|x_i - x_j|^{\Delta_i + \Delta_j}}. \tag{2}$$

The three point function is

$$\langle O_i O_j O_k \rangle = \frac{f_{ijk}}{|x_{ij}|^{\Delta_i + \Delta_j - \Delta_k} |x_{jk}|^{\Delta_j + \Delta_k - \Delta_i} |x_{ik}|^{\Delta_i + \Delta_k - \Delta_j}} \tag{3}$$

for some number f_{ijk} and $x_{ij} = x_i - x_j$. In a unitary theory the f_{ijk} are real. These coefficients match with those in the operator product expansion (OPE), which can be written as

$$O_i(x_i)O_j(x_j) = \sum_{O_k} C_{ijk}(x_{ik}, x_{jk}, \partial_k) O_k(x_k) \tag{4}$$

$$= \sum_{O_k} f_{ijk} C(x_{ik}, x_{jk}, \partial_k) O_k(x_k). \tag{5}$$

The sum runs over an orthonormal basis of primary operators, and C is a power series that generates contributions from the desecendent operators of the corresponding primary operator. It can be shown that for the OPE of two scalars only even spin representations contribute non-zero terms.

The four point function is the most interesting, and we will provide a more in depth analysis in a later section. Here, we claim that up to conformal transformations the location of four points can be specified by two conformally invariant cross ratios

$$u = \frac{x_{12}^2 x_{34}^2}{x_{13}^2 x_{24}^2} \qquad v = \frac{x_{23}^2 x_{14}^2}{x_{13}^2 x_{24}^2}. \tag{6}$$

For a single scalar ϕ we therefore have that

$$\langle \phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4) \rangle = \frac{g(u,v)}{|x_{12}|^{\Delta_\phi} |x_{34}|^{\Delta_\phi}}.$$

Permuting the x_i leads to two constraints on g:

$$g(u,v) = g(u/v, 1/v), \qquad g(u,v) = \left(\frac{u}{v}\right)^{\Delta_\phi} g(v,u) \tag{7}$$

the second of which is known as crossing symmetry. Plugging the OPE into the expression for the four point function and using an orthonormal basis of primaries gives the conformal block decomposition

$$g(u,v) = \sum_O f_{\phi\phi O}^2 g_{\Delta_O l_O}(u,v) \tag{8}$$

for known functions $g_{\Delta_O l_O}$ that are eigenfunctions of the quadratic casimir operator of the conformal group. They depend analytically on spacetime dimension and can be approximated well via series expansions in general d.

These results imply that one, two, three, and four point functions can all be determined with knowledge of operator content, dimensionality, and OPE coeffients. These quantities are referred to as CFT data. But not all values of scaling dimensions and OPE coefficients lead to a consistent CFT. Applying the crossing symmetry constraint gives

$$\sum_O f_{\phi\phi O}^2 (v^{\Delta_\phi} g_{\Delta_O l_O}(u,v) - u^{\Delta_\phi} g_{\Delta_O l_O}(v,u)) = 0. \tag{9}$$

Defining $F^{\Delta_\phi}_{\Delta_O, l_O}(u,v)$ as the quantity contained in the brackets allows us to interpret (9) as a sum of vectors (functions of u and v) with positive coefficients that gives zero. Contradicitons to situations such as the above exist when there exists a linear functional α that is positive on all the above vectors, and strictly positive on at least one. It turns out that for most sets of naively picked CFT data one can find a contradiction. In the case of the Ising model, this will allow us to rule out almost all choices of CFT data and determine critical exponents to high precision.

The most common choice for α is to choose z such that $u = z\bar{z}$ and $v = (1-z)(1-\bar{z})$ and set

$$\alpha(F) = \sum_{m+n \le N} a_{mn} \partial_z^m \partial_{\bar{z}}^m F(z, \bar{z})|_{z=\bar{z}=1/2}. \tag{10}$$

Plugging in different possible values for $\{\Delta_i, f_{ijk}\}$ and minimizing with respect to the parameters a_{mn} allows us to rule out large sectors of the space of conformal field theories.

II. BOOTSTRAPPING THE ISING MODEL

We review results on the application of this method to the Ising model. The Ising model is characterized by two relevant operators, the spin operator σ and the thermal operator ϵ. They are related to critical exponents via $\eta = 2\Delta_\sigma - d + 2$ and $\nu = 1/(d - \Delta_\epsilon)$.

Analyzing the Ising model with the conformal bootstrap started with the the four spin correlator $\langle\sigma\sigma\sigma\sigma\rangle$. Fixing Δ_σ, it is possible to get an upperbound for Δ_ϵ by demanding that $\alpha(F^{\Delta_\sigma}_{\Delta_\epsilon, 0}) \ge 0$ not have a solution for any a_{mn}. This can be used to construct an upper bound curve of Δ_ϵ as a function of Δ_σ [4].

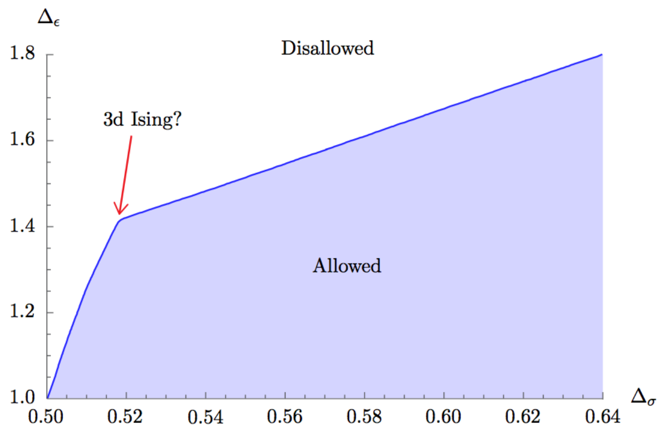

FIG. 1. Plot of the upperbound when applying the bootstrap to the correlator $\langle\sigma\sigma\sigma\sigma\rangle$, as described above. Taken from [4]. The kink on the boundary of the curve has operator dimensions that correspond to those of the 3d Ising model.

The upper bound curves for the Ising model at $d = 2, 3$ have the common feature that the correct critical exponents lie at the kink. This can be supported further by bootstrapping mixed correlators, and adding the condition that σ and ϵ are the only relevant operators. These conditions seem to be strong enough to pick out a single pair of scaling dimensions, and lead to the most precise determinantion of critical exponents for the 3d Ising model to date [2]. This powerful result seems to imply that the Ising model at criticality is the unique conformal field theory with two relevant operators, and that the numerical bootstrap approach is powerful enough to analyze it.

III. ANALYTICAL CONTINUATION TO FRACTIONAL DIMENSIONS

The conformal bootstrap approach has the advantage that, in most cases, it can be easily analytically continued to fractional dimensions, as d becomes only a parameter for which we can plug in arbitrary values. However, we shall see that this naive analytic continuation breaks down for $d < 2$.

To illustrate this, we examine in detail the four-point identical scalar function $\langle\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)\rangle$ given in [4]. Translational symmetry can be used to set $x_1 = 0$. Rotational symmetry and dialation symmetry can be used to set $x_3 = e_1$. Finally, invariance under special conformal transformations can be used to set $x_4 \to \infty$. This leaves x_2, which by rotational symmetry can be fixed to have the coordinates $(x, y, 0, ..., 0)$. Defining $z = x + iy$ and $u = z\bar{z}$ and $v = (1 - z)(1 - \bar{z})$ provides a definition for the crossratios and makes it clear why there are only two degrees of freedom in the four point function, despite there existing 4 coordinates in d dimensions.

This argument, however, breaks down in one dimension where there is only one degree of freedom for the placement of x_2. This forces z to be real, and gives a relation between the crossratios, now given by $v = (1 - x)^2$ and $u = x^2$. Functions of the crossratios are now required to satisfy $F(u, v) = g(x)$, lowering the number of constraints implied by crossing symmetry. This will be important to keep in mind when we analyze recent numerical results in $d < 2$.

For theories with $d \geq 2$, the number of independent crossratios is fixed at 2, and the only dependence on d lies in the conformal blocks, which are analytic in d. Therefore, it is trivial to analytically continue this procedure into non-integer $d \geq 2$: the calculations are the same as for d integer. In particular, this provides a non-perturbative definiton of conformal field theory in fractional dimension.

There is a problem, however. Recent studies in conformal perturbaion theory of the Wilson Fischer fixed point show that unitarity is violated even in the first order terms. Negative norm states were shown to appear, as well as operators with complex scaling dimensions [5]. Unitarity is critical to the bootstrap approach, as it ensures dimensions are real, bounded below as well as the positivitity of the squared OPE coefficients. However, unitary violation only occurs at high dimension: $\Delta \geq 23$. It is plausible that these small instances of unitarity violation at high dimension are sufficiently suppressed so that they do not impact the results of the bootstrap.

IV. INTERPRETATION OF NUMERICAL RESULTS IN FRACTIONAL DIMENSIONS

The bootstrap approach can now be performed in fractional dimension, and be compared to the epsilon expansion, for example. This was done in [3] and the results were found to be in strong agreement with the epsilon expansion. The scaling dimensions were taken from the kink on the boundary of the upper bound curve, as in two and three dimensions.

This technique can also be applied to $1 < d < 2$ by just replacing d in the expression for the conformal blocks. The situation here is much different here, however, as found in [1]. Their results, in contrast, immediately ruled out dimensions obtained from the curve found via the epsilon expansion as early as $d - 1.875$. Their results also disagreed with those found on fractal lattices, and they also did not capture the $d \to 1$ limit: their results for $d = 1.000001$ looked very different than the exact results in $d = 1$. Finally, instead of a single kink appearing where Ising scaling dimensions could be found, two kinks now appear for dimensions immediately below 2. The dimensions at both kinks disagree with what one would expect for the Ising model, now in the disallowed region.

Thus, for below $d = 2$, the bootstrap fails almost all checks for being consistent with the Ising model. There have been a couple potential explanations floated in the

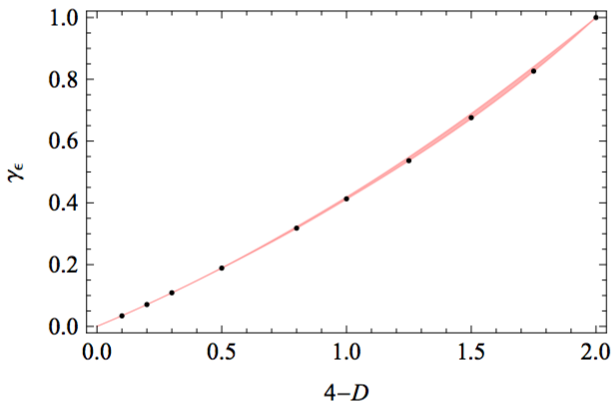

FIG. 2. Plot of the anomalous dimension of ϵ versus the deviation from the free theory in four dimensions. The black dots indicate bootstrap calculations whereas the red curve indicates results from the traditional epsilon expansion. Taken from [3].

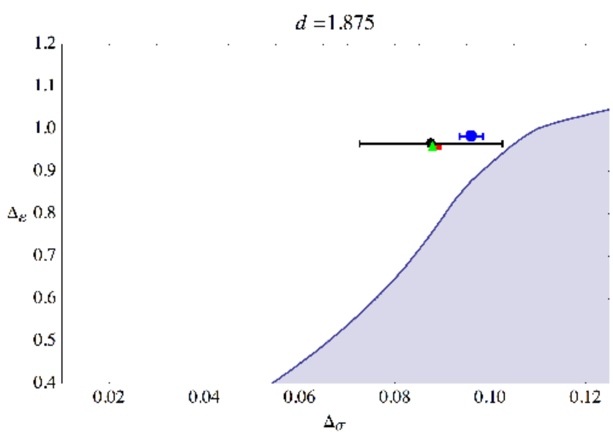

FIG. 3. Plot of the upper bound curve in $d = 1.875$. If one looks closely, one can see two kinks instead of one. The various predictions for Ising dimensions from other methods are plotted in other colors. All of them are ruled out. Taken from [1].

literature, most reducing to the claim that CFTs behave differently in $d = 2$ and that could correspond to sudden breaking of unitarity at low operator dimension for $d < 2$.

We propose a different explanation: the analytic continuation to fractional dimension is different in character for $d < 2$ because of the crossover between two independent cross ratios to one. In particular, we argue that spacial dimension has two effects on the bootstrap: the analytic effect on the conformal blocks as well as the determination of the number of cross ratios. The numerical calculations in [1] only accounted for the effect on the conformal blocks. We now argue that our proposal explains the odd numerical results.

The failure to capture the $d \to 1$ limit makes sense, since for any $d > 1$ there exist two independent crossratios in the numerical calculations. The jump from two crossratios in $d = 1.000001$ to one crossratio in $d = 1$ explains the failure of the two results to resemble eachother.

The sudden change of behavior at $d = 2$ also makes sense, as d only effects the number of independent cross ratios for $d \leq 2$.

We also can propose why two kinks show up instead of one. Our idea makes use of the fact that increasing the number of independent crossratios increases the number of constraints from crossing symmetry. The result is that by erroneously assuming that two independent crossratios exist, extra constraints appear. These could act as another upperbound, cutting off the kink that corresponds to the ising model. The process leaves two other kinks where the constraint intersects the upperbound curve.

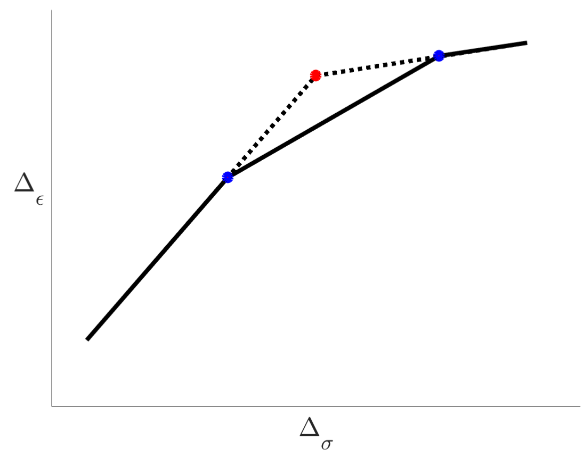

FIG. 4. Illustration of the above argument for why two kinks appear instead of one. The red dot corresponds to the Ising model, but it is removed by an extra constraint. This leaves behind two kinks, blue dots, neither of which resemble the Ising model in operator dimension.

We believe that this proposal has the potential to explain odd behavior in $1 < d < 2$. However, the best evidence will be a correct calculation that takes the changing number of independent crossratios into account quantitatively and naturally. We are not sure how to approach this for the non-perturbative bootstrap. Perhaps a full understanding of representations of the conformal group in fractional dimensions would allow for a better analysis, rather than a non-constructive approach like analytic continuation. Some open questions along these lines are pointed out in [5], and they may be worth investigating for this purpose.

V. CONFORMAL PERTURBATION THEORY

In this section, we present some thoughts on what a perturbative expansion around $d = 2$ would have to look like. This seems like a natural way to better understand, analyticaly, the behavior around $d = 2$ and may shed light on earlier difficulties for $d < 2$.

Such an expansion is different in character than that of the Wilson-Fischer fixed point near $d = 4$. For the Wilson-Fischer case, the free field fixed point in $d = 4$ splits into the free field fixed point in $d = 4 - \epsilon$ and the Wilson-Fischer fixed point. This comes about because the renormalization group flow is generated by a ϕ^4 perturbation that becomes relevant below $d = 4$.

The Ising model in $d = 2$, on the other hand, coincides with a free Majorana fermion [6]. But for above $d = 2$, it does not. The coincidence is the result fixed point collision, not a splitting due to the turning on of a relevant coupling. Therefore, we find it unlikely that any techniques using a local perturbation to the action for free Majorana fermions will succeed.

We therefore turn to conformal perturbation theory, and try to set up a calculation similar to [7]. Here, Rychkov and Tan calculate the first order correction in the epsilon expansion using only CFT methods. They did this by requiring that as $\epsilon \to 0$, all the operator dimensions and OPE coefficients approach those of the free field theory in $d = 4$. Except they also introduced another condition, that $\Delta_{\phi^3} = \Delta_\phi + 2$, coming from the equation of motion at the Wilson-Fischer fixed point. This allowed them to distinguish the Wislon-Fischer CFT from that of the free field.

We have the same problem for an expansion around $d = 2$ (how do we distinguish between free Majorana fermions in $d = 2 + \epsilon$ and the critical Ising model?) except we don't have an equation of motion that can help us. Instead, we propose the condition that the only two relevant operators are σ and ϵ, and no other operators receive perturbative corrections that would cause them to become relevant. As numerical results indicate that the Ising model is the unique $\mathbb{Z}_2$ symmetric CFT with only two relevant operators, we believe that in principle this condition is strong enough to specify $O(\epsilon)$ corrections around $d = 2$. We hope to pursue further steps in this calculation in future work.

VI. CONCLUSIONS

In this work, we reviewed the conformal bootstrap approach to the Ising model. We examined technical aspects of the analytic continuation to fractional dimensions, and proposed that not considering the effects on the number on independent crossratios spoiled numerical results for $1 < d < 2$. We took a detailed look at the numerical results, and provided potential explanations for each major feature using our proposal. Finally, we talked about how one might be able to gain analytic control of the $d \approx 2$ regime using CFT methods, despite the failure of Lagrangian ones. It is an open question how to properly define and/or understand CFTs in $1 < d < 2$, and future work in this direction should be very interesting.

ACKNOWLEDGMENTS

I would like to thank Professor Mehran Kardar and Alex Siegenfeld for their instruction in 8.334, the class that prompted this research. The class was challenging, but incredibly rewarding due to their effort in providing clear explanations for the complicated concepts involved. I would also like to thank Henry Shackleton for the countless conversations throughout the semester regarding classwork, as well as helpful discussions about the work done in our projects.

[1] J. Golden and M. F. Paulos, JHEP **03**, 167 (2015), arXiv:1411.7932 [hep-th].
[2] F. Kos, D. Poland, and D. Simmons-Duffin, JHEP **11**, 109 (2014), arXiv:1406.4858 [hep-th].
[3] S. El-Showk, M. Paulos, D. Poland, S. Rychkov, D. Simmons-Duffin, and A. Vichi, Phys. Rev. Lett. **112**, 141601 (2014), arXiv:1309.5089 [hep-th].
[4] D. Simmons-Duffin, in *Proceedings, Theoretical Advanced Study Institute in Elementary Particle Physics: New Frontiers in Fields and Strings (TASI 2015): Boulder, CO, USA, June 1-26, 2015* (2017) pp. 1–74, arXiv:1602.07982 [hep-th].
[5] M. Hogervorst, S. Rychkov, and B. C. van Rees, Phys. Rev. **D93**, 125025 (2016), arXiv:1512.00013 [hep-th].
[6] P. D. Francesco, P. Mathieu, and D. Sanachal, *Conformal Field Theory* (Springer, 1997).
[7] S. Rychkov and Z. M. Tan, J. Phys. **A48**, 29FT01 (2015), arXiv:1505.00963 [hep-th].