The Nishimori Line in the Random Bond Ising Model

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The random bond Ising model (RBIM) serves as a key example of an inhomogenous system with nontrivial phase transitions. An important aid to our theoretical understanding of the RBIM is the Nishimori line, which is derived from a gauge transformation of the RBIM that allows for the exact calcuation of certain physical properties along a particular manifold. We review the results of this discovery and describe several applications of the Nishimori line in the two-dimensional RBIM.

INTRODUCTION

Ising-type models play an important role as the simplest nontrivial models for describing magnetically ordered phases and corresponding phase transitions. For these models, exact results such as Onsager's 1944 solution for the two-dimensional Ising model [1] as well as those due to duality transformations [2] have provided important theoretical anchors and useful tools for computational efforts. We will focus on the random bond Ising model (RBIM), which serves as a simple model for phase transitions in a system with quenched disorder. An experimental example of such a system is the transition between plateaus in the integer quantum Hall effect [3]. The discovery of the Nishimori line similarly provides a theoretical tool for analyzing the phase diagram of the RBIM. In particular, along the Nishimori line, we can obtain an exact expression for the internal energy, an upper bound for the specific heat, and rigorous relations between correlation functions [4]. This insight is an important step in understanding the multicritical point in the RBIM phase diagram (Fig. 1).

Model and Phase Diagram

The random bond Ising model (RBIM) consists of the standard Ising Hamiltonian with nearest neighbor couplings

$$\beta H = -\sum_{\langle i,j \rangle} K_{ij} S_i S_j, \tag{1}$$

where S_i, S_j are Ising variables that take on values ± 1, and K_{ij} is a random coupling given by the probability distribution $P(K_{ij})$. In particular, consider the RBIM on a square lattice with a binary probability distribution

$$P(K_{ij}) = p\,\delta(K_{ij} - K) + (1-p)\,\delta(K_{ij} + K), \tag{2}$$

with $K > 0$. Because we can map $p \to 1 - p$ while flipping spins on one sublattice to map the antiferromagnetic to the ferromagnetic transition, we only show the phase diagram for $p > 1/2$ (Fig. 1). At $p = 1$, we have the standard ferromagnetically coupled two-dimensional Ising model with $T_c^{-1} = K_c = \ln\left(\sqrt{2}+1\right)/2 \approx 0.44$. For $p < 1$, this critical temperature is reduced by frustration induced by the bond configuration until it vanishes at $p = p_c \approx 0.88$ [5]. Unlike in higher dimensions, the two-dimensional RBIM does not have a spin glass phase at finite temperature [6–8].

The Nishimori line, shown as a dotted line in Fig. 1, is not a phase boundary, but a manifold on which a gauge symmetry allows for a series of exact results. In addition, the Nishimori line has the property of being invariant under RG transformations. More precisely, the property that the distribution $P(K_{ij})$ is on the Nishimori line is preserved under RG [9]. Therefore, the point at which the Nishimori line crosses the phase boundary, which is also stable under RG, is a nontrivial multicritical fixed point (N). The properties of the Nishimori line will be discussed in detail below.

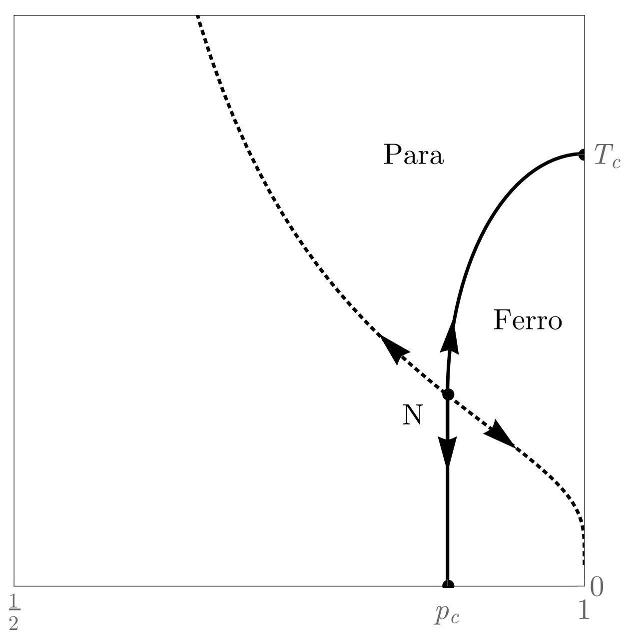

FIG. 1. Phase diagram (p vs. $T = K^{-1}$) for the 2D RBIM on a square lattice with fixed points, phase boundaries (solid line), and the Nishimori line (dotted line). Ferromagnetic and paramagnetic phases are labeled along with the multicritical point (N). Arrows show the RG flow near N.

NISHIMORI LINE

Starting with the Ising Hamiltonian (1), the quenched free energy is

$$-[\beta F] = \int_{-\infty}^{\infty} \prod_{\langle i,j\rangle} P(K_{ij})\, dK_{ij}\, \ln\left(\sum_{\{S_i\}} e^{\sum_{\langle i,j\rangle} K_{ij} S_i S_j}\right),$$

where $[\cdot]$ represents the quenched average over systems with bond distribution $P(K_{ij})$. We can now rewrite the quenched average by replacing our coupling $K_{ij} \to \tau_{ij}\,|K_{ij}|$, where we have introduced a new variable $\tau_{ij} = \pm 1$. From now on, we will drop the absolute value signs and simply refer to the new positive coupling as $K_{ij} = |K_{ij}| > 0$. The equivalent probability distribution can be written as

$$P(K_{ij}, \tau_{ij}) = (P(K_{ij}) + P(-K_{ij}))\, \frac{e^{\tau_{ij}\tilde{K}_{ij}}}{2\cosh \tilde{K}_{ij}}, \tag{3}$$

where $\exp(-2\tilde{K}_{ij}) \equiv P(-K_{ij})/P(K_{ij})$. We can write the partition function

$$Z(\{\tau_{ij}\}, \{K_{ij}\}) = \sum_{\{S_i\}} e^{\sum_{\langle i,j\rangle} \tau_{ij} K_{ij} S_i S_j}. \tag{4}$$

Then, the free energy is then given by

$$\begin{aligned} -[\beta F] &= \int_0^{\infty} \prod_{\langle i,j\rangle} dK_{ij} \sum_{\{\tau_{ij}\}} \prod_{\langle i,j\rangle} P(K_{ij}, \tau_{ij}) \ln Z(\{\tau_{ij}\}, \{K_{ij}\}) \\ &= \int_0^{\infty} \prod_{\langle i,j\rangle} dK_{ij} \left(\prod_{\langle i,j\rangle} \frac{P(K_{ij}) + P(-K_{ij})}{2\cosh \tilde{K}_{ij}}\right) \sum_{\{\tau_{ij}\}} e^{\sum_{\langle i,j\rangle} \tau_{ij}\tilde{K}_{ij}} \ln Z(\{\tau_{ij}\}, \{K_{ij}\}), \end{aligned} \tag{5}$$

Gauge Transformation

We can now perform a local $\mathbb{Z}_2$ gauge transformation through a new set of Ising variables σ_i, where we map $S_i \to S_i\sigma_i$ and $\tau_{ij} \to \tau_{ij}\sigma_i\sigma_j$. This transformation leaves $Z(\{\tau_{ij}\}, \{K_{ij}\})$ unchanged and does not alter the quenched average since we are summing over all $\tau_{ij} = \pm 1$ and $S_i = \pm 1$. Furthermore, we can average over all such gauge transformations, giving the following expression for the quenched free energy

$$\begin{aligned} -[\beta F] &= 2^{-N} \int_0^{\infty} \prod_{\langle i,j\rangle} dK_{ij} \left(\prod_{\langle i,j\rangle} \frac{P(K_{ij}) + P(-K_{ij})}{2\cosh \tilde{K}_{ij}}\right) \sum_{\{\tau_{ij}\}} \sum_{\{\sigma_i\}} e^{\sum_{\langle i,j\rangle} \tau_{ij}\tilde{K}_{ij}\sigma_i\sigma_j} \ln Z(\{\tau_{ij}\}, \{K_{ij}\}) \\ &= 2^{-N} \int_0^{\infty} \prod_{\langle i,j\rangle} dK_{ij} \left(\prod_{\langle i,j\rangle} \frac{P(K_{ij}) + P(-K_{ij})}{2\cosh \tilde{K}_{ij}}\right) \sum_{\{\tau_{ij}\}} Z(\{\tau_{ij}\}, \{\tilde{K}_{ij}\}) \ln Z(\{\tau_{ij}\}, \{K_{ij}\}), \end{aligned} \tag{6}$$

where N is the number of sites. In the above expression, we have also noted that $Z(\{\tau_{ij}\}, \{\tilde{K}_{ij}\}) = \sum_{\{\sigma_i\}} \exp\left(\sum_{\langle i,j\rangle} \tau_{ij}\tilde{K}_{ij}\sigma_i\sigma_j\right)$. This identification implies that the weight of a particular distribution of frustration is proportional to the partition function of an Ising model with effective couplings $\tilde{K}_{ij}$. This will be an important observation when discussing shape of the phase boundary in a later section.

Given this form of the free energy (6), we define the Nishimori manifold by the condition

$$e^{-2K_{ij}} = e^{-2\tilde{K}_{ij}} \equiv \frac{P(-K_{ij})}{P(K_{ij})}. \tag{7}$$

Note, however, that this condition cannot be satisfied by any general probability distribution. In fact, the distribution must be of the form

$$P(K_{ij}) = e^{K_{ij}} f(K_{ij}), \tag{8}$$

where $f(K_{ij}) = f(-K_{ij})$ is even. For the binary distribution (2), the condition (7) is met if

$$e^{-2K} = e^{-2\tilde{K}} \equiv \frac{1-p}{p}. \tag{9}$$

This constraint gives the Nishimori line shown in the above phase diagram (Fig. 1).

APPLICATIONS

For the following results, we will begin with the general RBIM with $P(K_{ij})$ satisfying the Nishimori condition (7) and then specialize to the two-dimensional RBIM with a binary distribution on a square lattice.

Internal Energy

Assuming the Nishimori condition (7), we can compute the internal energy by taking a derivative of the free energy (where $K_{ij} \propto \beta$) and then take the quenched average as shown above (6). This gives us

$$\begin{aligned}
[\langle E \rangle] = -\left[\frac{\partial(-\beta F)}{\partial \beta}\right] &= -2^{-N} \int_0^\infty \prod_{\langle i,j \rangle} dK_{ij} \left(\prod_{\langle i,j \rangle} \frac{P(K_{ij}) + P(-K_{ij})}{2\cosh K_{ij}} \right) \sum_{\{\tau_{ij}\}} Z(\{\tau_{ij}\},\{K_{ij}\}) \frac{\partial}{\partial \beta} \ln Z(\{\tau_{ij}\},\{K_{ij}\}) \\
&= -2^{-N} \int_0^\infty \prod_{\langle i,j \rangle} dK_{ij} \left(\prod_{\langle i,j \rangle} \frac{P(K_{ij}) + P(-K_{ij})}{2\cosh K_{ij}} \right) \sum_{\{\tau_{ij}\}} \frac{\partial}{\partial \beta} Z(\{\tau_{ij}\},\{K_{ij}\}) \\
&= -2^{-N} \int_0^\infty \prod_{\langle i,j \rangle} dK_{ij} \left(\prod_{\langle i,j \rangle} \frac{P(K_{ij}) + P(-K_{ij})}{2\cosh K_{ij}} \right) \sum_{\{S_i\}} \sum_{\{\tau_{ij}\}} \frac{\partial}{\partial \beta} e^{\sum_{\langle i,j \rangle} \tau_{ij} K_{ij} S_i S_j} \\
&= -\sum_{\langle i,j \rangle} \int_0^\infty dK_{ij} \, \left(P(K_{ij}) + P(-K_{ij})\right) K_{ij} \tanh\left(K_{ij}\right),
\end{aligned} \tag{10}$$

with $\langle \cdot \rangle$ denoting the thermal average. We can use an equivalent form of the Nishimori condition

$$\tanh K_{ij} = \frac{P(K_{ij}) - P(-K_{ij})}{P(K_{ij}) + P(-K_{ij})}, \tag{11}$$

to further simplify this result. The internal energy reduces to

$$\begin{aligned}
[\langle E \rangle] &= -\sum_{\langle i,j \rangle} \int_{-\infty}^\infty dK_{ij} \, P(K_{ij}) K_{ij} \\
&= -N_B \left[K_{ij}\right],
\end{aligned} \tag{12}$$

where N_B is the number of bonds. For the binary distribution (2) on the square lattice,

$$[\langle E \rangle] = -2NK(2p-1) = -2NK \tanh K, \tag{13}$$

where $2p - 1 = \tanh K$ is fixed on the Nishimori line (9). Note that this internal energy is analytic even though the Nishimori line crosses the phase boundary (Fig. 1). As Nishimori notes in his original paper, this is not contradictory since the absence of a singularity in the internal energy does not preclude singularities from occuring in other physical quantities [4].

Heat Capacity

Another quantity that we can investigate by taking advantage of the special properities of the Nishimori line is the heat capacity. Although we cannot compute the heat capacity exactly, we can determine an upper bound using Jensen's inequality $\left[\langle E \rangle^2\right] \geq [\langle E \rangle]^2$. Therefore,

$$\begin{aligned}
[\langle C \rangle] &= \left[\langle E^2 \rangle\right] - \left[\langle E \rangle^2\right] \\
&\leq \left[\langle E^2 \rangle\right] - [\langle E \rangle]^2 \\
&= N_B \left(\left[K_{ij}^2\right] - [K_{ij}]^2 \right),
\end{aligned} \tag{14}$$

where $\left[\langle E^2 \rangle\right]$ is computed using the same method as for the internal energy (10) but replacing $\partial/\partial\beta \to \partial^2/\partial\beta^2$. For the binary distribution on the square lattice, this gives the bound

$$[\langle C \rangle] \leq 8NK^2 p(1-p) = 2NK^2 \operatorname{sech}^2 K. \tag{15}$$

This bound implies that the heat capacity is finite along the Nishimori line, so the critical exponent $\alpha < 0$ at the multicritical point N where the Nishimori line crosses the phase boundary (Fig. 1).

Correlation Functions

We can use the same technique applied above for internal energy and heat capacity to prove some general results about correlation functions. Consider a correlation function $\left[\langle S^X \rangle\right]$ for an arbitrary product of spins S^X. Without assuming we are on the Nishimori manifold, we first write the quenched average of the correlation function analogously with the free energy expression (5). This gives

$$[\langle S^X \rangle] = \int_0^\infty \prod_{\langle i,j \rangle} dK_{ij} \left(\prod_{\langle i,j \rangle} \frac{P(K_{ij}) + P(-K_{ij})}{2\cosh \tilde{K}_{ij}} \right) \sum_{\{\tau_{ij}\}} e^{\sum_{\langle i,j \rangle} \tau_{ij} \tilde{K}_{ij}} \langle S^X \rangle . \tag{16}$$

Again, because we are summing over all $\tau_{ij} = \pm 1$ and $S_i = \pm 1$, a gauge transform will not alter this quenched average, so we can now perform an average over all gauge transformations. Under the gauge transformation $\langle S^X \rangle \to \sigma^X \langle S^X \rangle$, where σ^X is the same product of spins as S^X but with all S_i replaced by σ_i. Therefore,

$$\begin{aligned} [\langle S^X \rangle] &= 2^{-N} \int_0^\infty \prod_{\langle i,j \rangle} dK_{ij} \left(\prod_{\langle i,j \rangle} \frac{P(K_{ij}) + P(-K_{ij})}{2\cosh \tilde{K}_{ij}} \right) \sum_{\{\tau_{ij}\}} \sum_{\{\sigma_i\}} \sigma^X e^{\sum_{\langle i,j \rangle} \tau_{ij} \tilde{K}_{ij} \sigma_i \sigma_j} \langle S^X \rangle \\ &= 2^{-N} \int_0^\infty \prod_{\langle i,j \rangle} dK_{ij} \left(\prod_{\langle i,j \rangle} \frac{P(K_{ij}) + P(-K_{ij})}{2\cosh \tilde{K}_{ij}} \right) \sum_{\{\tau_{ij}\}} \sum_{\{\sigma_i\}} e^{\sum_{\langle i,j \rangle} \tau_{ij} \tilde{K}_{ij} \sigma_i \sigma_j} \langle S^X \rangle_{\tilde{K}} \langle S^X \rangle_K , \end{aligned} \tag{17}$$

where we replaced the term $\sum_{\{\sigma_i\}} \sigma^X e^{\sum_{\langle i,j \rangle} \tau_{ij} \tilde{K}_{ij} \sigma_i \sigma_j} \to \sum_{\{\sigma_i\}} e^{\sum_{\langle i,j \rangle} \tau_{ij} \tilde{K}_{ij} \sigma_i \sigma_j} \langle \sigma^X \rangle_{\tilde{K}}$ and relabeled the Ising variable in thermal average $\langle \sigma^X \rangle_{\tilde{K}} \to \langle S^X \rangle_{\tilde{K}}$. Here, the subscripts on the thermal averages refer to the Ising interaction over which we are performing the thermal average. As a last step, we invert the average over gauge transformations, noting that the two thermal averages $\langle S^X \rangle_{\tilde{K}} \langle S^X \rangle_K$ have factors that will cancel out under a gauge transformation. This brings the expression back into the familiar form

$$[\langle S^X \rangle_K] = \int_0^\infty \prod_{\langle i,j \rangle} dK_{ij} \left(\prod_{\langle i,j \rangle} \frac{P(K_{ij}) + P(-K_{ij})}{2\cosh \tilde{K}_{ij}} \right) \sum_{\{\tau_{ij}\}} \sum_{\{\sigma_i\}} e^{\sum_{\langle i,j \rangle} \tau_{ij} \tilde{K}_{ij}} \langle S^X \rangle_{\tilde{K}} \langle S^X \rangle_K . \tag{18}$$

We recognize this as our original expression for the quenched average, analogous to (5) and (16), and can write our result as

$$[\langle S^X \rangle_K] = [\langle S^X \rangle_{\tilde{K}} \langle S^X \rangle_K] , \tag{19}$$

recalling that we defined the interaction $\tilde{K}_{ij}$ as a function of K_{ij} by $\exp(-2\tilde{K}_{ij}) \equiv P(-K_{ij})/P(K_{ij})$. If the system satisfies the Nishimori condition $\tilde{K}_{ij} = K_{ij}$, then the result reduces to

$$[\langle S^X \rangle] = \left[\langle S^X \rangle^2 \right] \geq 0. \tag{20}$$

Furthermore, because of the invariance of even powers of the thermal average $\langle S^X \rangle$ under gauge invariance, we can generalize this relation between correlations functions using the same method to obtain

$$\left[\langle S^X \rangle^{2k+1} \right] = \left[\langle S^X \rangle^{2k+2} \right] \geq 0, \tag{21}$$

for $k \geq 0$. This result provides further insight into the phase diagram of the general RBIM. Specifically, in the spin glass phase, we expect

$$\lim_{|i-j| \to \infty} \left[\langle S_i S_j \rangle^{2k+1} \right] = 0 \text{ and } \lim_{|i-j| \to \infty} \left[\langle S_i S_j \rangle^{2k+2} \right] \neq 0, \tag{22}$$

which is incompatible with the previous result, implying that the Nishimori manifold cannot pass through any spin glass phase [9]. Furthermore, because of this equality (21) along with the RG analysis described in the introduction [9], we can place the multicritical point—where the spin glass (if it exists), paramagnetic, and ferromagnetic phases meet—on the Nishimori line for the RBIM with a binary distribution. This result is consistent with the ϵ expansion, which confirms the scaling (shown in Fig. 1) near the multicritical point N [10].

Phase Boundary Shape

In the particular case of the RBIM with a binary distribution (2), we can write the free energy (6) in the simplified form

$$-[\beta F] = 2^{-N}\left(2\cosh\tilde{K}\right)^{-N_B}\sum_{\{\tau_{ij}\}} Z(\{\tau_{ij}\},\tilde{K})\ln Z(\{\tau_{ij}\},K). \tag{23}$$

In this form, we explicitly see that the weight of a particular distribution of $\{\tau_{ij}\}$, i.e. distribution of frustration, is given by the partition function $Z(\{\tau_{ij}\},\tilde{K})$. Along the Nishimori line, this becomes

$$-[\beta F] = 2^{-N}\left(2\cosh\tilde{K}\right)^{-N_B}\sum_{\{\tau_{ij}\}} Z(\{\tau_{ij}\},\tilde{K})\ln Z(\{\tau_{ij}\},\tilde{K}). \tag{24}$$

with $K = \tilde{K}$ and p related by the Nishimori condition (9). Nishimori then reinterpreted this free energy as the entropy of frustration, using the indentification of the partition function with the probability weight of frustration [11]. Since the Nishimori line passes through the multicritical point N, we know that at N, where $p = p_c^*$, the free energy (24) is singular. Therefore, the entropy of frustration is also singular at $p = p_c^*$. However, this entropy of frustration is independent of temperature, only dependent on geometry. Therefore, we can identify $p_c^* = p_c$ (the critical probability at zero temperature). Moreover, this singularity in the entropy of frustration implies the phase boundary below the multicritical point in the phase diagram (Fig. 1) is vertical, since the phase transition is induced by geometry.

CONCLUSION

We have reviewed the key results of Nishimori [4], which have been extended and generalized [7, 12, 13] and have aided in numerical work on the RBIM [6, 14]. As a key feature in the phase diagram of the RBIM that passes through a multicritical fixed point, the Nishimori line and its unique features provide an excellent tool for studying this novel critical point. The Nishimori line also serves as an example of the exact results that can be obtained using gauge transformation methods in random models.

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