The Ising Model as Fermionic Field Theories

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The high temperature picture of the Ising model is a sum over loops in 2D and surfaces in 3D. However, free loop/surface theories have an overcounting problem due to self-intersection of oriented objects. Thus, a fermionic structure can be realized to remove overcounting, allowing us to write a fermionic representation of the 2D and 3D Ising model. In 2D, this manifests as a free majorana field theory and in 3D, it is hypothesized to become a fermionic string theory. A standard method for deriving the 2D fermionic Ising theory is presented, with a discussion of the critical point of the fermionic representation coinciding with massless fermions. The conformal theory at criticality is then explored for the 2D case, and extended to the 3D case for the hypothetical fermionic string theory.

I. INTRODUCTION

The 2D Ising model was analytically solved by Onsager using a transfer matrix formalism, deriving his famous result for the free energy of a rectangular 2D Ising model. [1] After Berezin developed the integration of anticommuting Grassmann variables, he realized it could be used to simplify Onsager's transfer matrix method. [2] Since then, both the 2D and 3D Ising models near criticality have been successfuly described by fermionic field theories. [3, 4]

Modern study of critical exponents has shifted towards investigating the conformal field theory at criticality. [5] The Ising-fermion relationships in two and three dimensions provide a powerful tool for calculating the critical exponents by finding the scaling dimensions of the spin and energy density operators.

II. GRASSMANN VARIABLES

To start, we introduce the calculus of Grassmann variables. As anticommuting variables, two Grassmann variables η_i and η_j must satisfy

$$\eta_i\eta_j = -\eta_j\eta_i$$

the main consequence of which is that $\eta^2 = 0$. Additionally, any function of Grassmann variables $f(\{\eta_i\})$ has the property $f(\{\eta_i\})^2 = 0$. [6]

The fundamental difference between Grassmann variables and commuting variables is that Grassmannian integration acts identically to differentiation. [6] Namely

$$\int \mathrm{d}\eta\, 1 = 0$$
$$\int \mathrm{d}\eta\, \eta = 1$$

From anticommutivity, we see that $\exp[f(\eta)] = 1 + f(\eta)$. Thus, Gaussian integrals over Grassmann variables take a particularly simple form

$$\int \mathrm{d}\bar{\eta}\, \mathrm{d}\eta\, e^{-\bar{\eta}A\eta} = A \tag{1}$$

and more generally

$$\int \mathrm{d}\bar{\eta}\, \mathrm{d}\eta\, e^{-\bar{\eta}_i A_{ij}\eta_j} = \det(A) \tag{2}$$

III. HIGH TEMPERATURE EXPANSION OF 2D ISING MODEL

Plechko considers a generic anisotropic triangular lattice, with hamiltonian [3]

$$\begin{aligned} -\beta\mathcal{H}[\{\sigma\}] = \sum_{mn} & \beta J_1 \sigma_{m,n}\sigma_{m+1,n} \\ & + \beta J_3\sigma_{m,n}\sigma_{m+1,n+1} + \beta J_1 \sigma_{mn}\sigma_{m+1n} \end{aligned} \tag{3}$$

where $\sigma = \pm 1$.

Using the fact that $\sigma^2 = 1$, we can write

$$\begin{aligned} e^{\beta J_i \sigma_a \sigma_b} &= \cosh(\beta J_i) + \sinh(\beta J_i)\sigma_a\sigma_b \\ &= \cosh(\beta J_i)[1 + t_i\sigma_a\sigma_b] \end{aligned}$$

where $t_i = \tanh(\beta J_i)$

With the following substitutions

$$(\sigma_1, \sigma_2, \sigma_2)_{mn} = (\sigma_{m,n}, \sigma_{m+1,n}, \sigma_{m+1,n+1})$$

$$\begin{aligned} \alpha_0 &= 1 + t_1t_2t_3 \\ \alpha_1 &= t_1 + t_2t_3 \\ \alpha_2 &= t_2 + t_1t_3 \\ \alpha_1 &= t_3 + t_1t_2 \end{aligned}$$

we can rewrite the partition function as [3]

$$Z = [2\cosh(\beta J_1)\cosh(\beta J_2)\cosh(\beta J_3)]^N Q$$

$$Q = \mathrm{Sp}_\sigma \prod_{mn} \left[\alpha_0 + \alpha_1\sigma_1\sigma_2 + \alpha_2\sigma_2\sigma_3 + \alpha_3\sigma_1\sigma_3\right]_{mn} \tag{4}$$

where $\mathrm{Sp}[\cdots]$ is the normalized spin average.

IV. THE 2D ISING MODEL AS A FREE MAJORANA THEORY

We now aim to include Grassmann variables and eliminate the spin variables. Consider the equality

$$\int \mathrm{d}\bar{\eta}\,\mathrm{d}\eta\,[1+(\alpha_1\sigma_1\sigma_2+\alpha_2\sigma_2\sigma_3+\alpha_3\sigma_1\sigma_3)\eta\bar{\eta}] = \int \mathrm{d}\bar{\eta}\,\mathrm{d}\eta\,\Big(1+\eta\frac{\alpha_1}{\sqrt{\chi}}\sigma_1\Big)\Big(1+\sqrt{\chi}(\eta+\bar{\eta})\sigma_2\Big)\Big(1+\bar{\eta}\frac{\alpha_2}{\sqrt{\chi}}\sigma_3\Big)$$

with $\chi = \alpha_1\alpha_2/\alpha_3$

Using (1), Q can be written as [7]

$$\begin{aligned}
Q &= \mathrm{Sp}_\sigma \prod_{mn}\int \mathrm{d}\bar{\eta}\,\mathrm{d}\eta\,\exp\Big\{\Big[\alpha_0+\alpha_1\sigma_1\sigma_2+\alpha_2\sigma_2\sigma_3+\alpha_3\sigma_1\sigma_3\Big]_{mn}\eta\bar{\eta}\Big\}\\
&= \mathrm{Sp}_\sigma \prod_{mn}\int \mathrm{d}\bar{\eta}\,\mathrm{d}\eta\,e^{\alpha_0\eta\bar{\eta}}\Big(1+\eta\frac{\alpha_1}{\sqrt{\chi}}\sigma_1\Big)\Big(1+\sqrt{\chi}(\eta+\bar{\eta})\sigma_2\Big)\Big(1+\bar{\eta}\frac{\alpha_2}{\sqrt{\chi}}\sigma_3\Big)\\
&= \mathrm{Sp}_\sigma \prod_{mn}\int \mathrm{d}\bar{\eta}\,\mathrm{d}\eta\,e^{\alpha_0\eta\bar{\eta}} B^{(1)}_{m,n}B^{(2)}_{m+1,n}B^{(3)}_{m+1,n+1}\\
&= \int\prod_{mn}\mathrm{d}\bar{\eta}_{mn}\,\mathrm{d}\eta_{mn}\,e^{\alpha_0\eta_{mn}\bar{\eta}_{mn}}\,\mathrm{Sp}_\sigma\prod_{n}^{L}\Big[B^{(3)}_{L,n}\dots B^{(3)}_{1,n}\times\Big(B^{(2)}_{1,n}B^{(1)}_{1,n}\Big)\dots\Big(B^{(2)}_{L,n}B^{(1)}_{L,n}\Big)\Big]
\end{aligned}$$

where in the last line we use mirror factorization to reorder the $B^{(i)}_{mn}$ factors.

We then spin average $B^{(3)}_{m,n}B^{(2)}_{m,n}B^{(1)}_{m,n}$ for each (m,n) (which equals a commuting exponential), reducing Q to a fully Grassmannian expression. [7]

$$\begin{aligned}
Q = \int\prod_{mn}\mathrm{d}\bar{\eta}_{mn}\,\mathrm{d}\eta_{mn}\,\exp\Big\{\sum_{mn}\Big[&(\alpha_0\eta_{m,n}\bar{\eta}_{m,n}-\alpha_1\eta_{m,n}\bar{\eta}_{m-1,n}-\alpha_2\eta_{m,n}\bar{\eta}_{m,n-1}-\alpha_3\eta_{m,n}\bar{\eta}_{m-1,n-1})\\
&-\alpha_1\eta_{m,n}\eta_{m-1,n}-\alpha_2\bar{\eta}_{m,n}\bar{\eta}_{m,n-1}\Big]\Big\}
\end{aligned} \tag{5}$$

We introduce the discrete derivatives $\partial_1\eta_{m,n} = \eta_{m,n}-\eta_{m-1,n}$ and $\partial_2\eta_{m,n} = \eta_{m,n}-\eta_{m,n-1}$, and define fermionic fields $\psi(m,n)=\eta_{mn}$ and $\bar{\psi}(m,n)=\bar{\eta}_{mn}$. Thus, in the continuum limit we can write (5) as a fermionic field theory [3]

$$\begin{aligned}
Q = \int \mathrm{D}\bar{\psi}\mathrm{D}\psi\,\exp\Big\{\int \mathrm{d}^2x\,\Big[&\underline{m}\psi(x)\bar{\psi}(x)+\lambda_1\psi(x)\partial_1\bar{\psi}(x)+\lambda_2\psi(x)\partial_2\bar{\psi}(x)\\
&-\alpha_3\psi(x)\partial_1\partial_2\bar{\psi}(x)+\alpha_1\psi(x)\partial_1\psi(x)+\alpha_2\bar{\psi}(x)\partial_2\bar{\psi}(x)
\end{aligned} \tag{6}$$

$$\underline{m} = \alpha_0-\alpha_1-\alpha_2-\alpha_3$$
$$\lambda_1 = \alpha_1+\alpha_3 \quad \& \quad \lambda_2 = \alpha_2+\alpha_3$$

Ignoring the second order kinetic term $\alpha_3\psi(x)\partial_1\partial_2\bar{\psi}(x)$, by an appropriate linear transformation of $\psi(x)$ and $\bar{\psi}(x)$ and a rescaling of $\mathbf{x}$, (6) can be written as a free majorana theory [3]

$$\begin{aligned}
Q &= \int \mathrm{D}\bar{\Psi}\mathrm{D}\Psi\,\exp\Big\{\int \mathrm{d}^2x\,\Big[\overline{m}\bar{\Psi}(x)\Psi(x)+\bar{\Psi}(x)\partial\bar{\Psi}(x)+\Psi(x)\bar{\partial}\Psi(x)\Big]\Big\}\\
&= \int \mathrm{D}\bar{\Psi}\mathrm{D}\Psi\,\exp\Bigg\{\int \mathrm{d}^2x\,\Bigg[\begin{pmatrix}\bar{\Psi}\\ \Psi\end{pmatrix}^T\Big(i\sigma_2\Big)\Big(\overline{m}+\sigma_1\partial_1+\sigma_2\partial_2\Big)\begin{pmatrix}\bar{\Psi}\\ \Psi\end{pmatrix}\Bigg]\Bigg\}
\end{aligned} \tag{7}$$

$$\overline{m} = \frac{\alpha_0-\alpha_1-\alpha_2-\alpha_3}{(2\sqrt{\alpha_0\alpha_1\alpha_2\alpha_3})^{1/2}}$$
$$\partial = \frac{1}{2}(\partial_1+i\partial_2)$$
$$\bar{\partial} = \frac{1}{2}(\partial_1-i\partial_2)$$

V. CRITICALITY OF 2D ISING MODEL

By Fourier transforming (5) and integrating with (2), one arrives at exactly the Onsager expression for a triangular lattice [3]

$$-\beta f_Q = \frac{1}{2}\int\int \frac{\mathrm{d}q}{2\pi}\frac{\mathrm{d}q'}{2\pi} \ln\Big[(\alpha_0^2+\alpha_1^2+\alpha_2^2+\alpha_3^2) - 2(\alpha_0\alpha_1 - \alpha_2\alpha_3)\cos(q) - 2(\alpha_0\alpha_2 - \alpha_1\alpha_3)\cos(q') - 2(\alpha_0\alpha_3 - \alpha_1\alpha_2)\cos(q+q')\Big] \tag{8}$$

Expanding to quadratic order in q and q' and rescaling the momenta, we get the singular part of the free energy as

$$\begin{aligned} -\beta f_{sing} &= \frac{1}{8\pi^2}\int \mathrm{d}^2 q \ln\left[\overline{\alpha}^2 + q^2\right] \\ &= \frac{1}{8\pi}\overline{\alpha}^2 \ln\frac{\text{const}}{\overline{\alpha}^2} \end{aligned} \tag{9}$$

$$\overline{\alpha} = \frac{\alpha_0 - \alpha_1 - \alpha_2 - \alpha_3}{(2\sqrt{\alpha_0\alpha_1\alpha_2\alpha_3})^{1/2}} \equiv \overline{m}$$

Thus, the mass $\overline{m}$ of the fermionic theory is proportional to $|Tc - T|$, and the fermions become massless at criticality.

VI. 2D CONFORMAL FREE FERMION THEORY

Because a field theory becomes conformally invariant at criticality, the Ising-fermion relationship gives us another method of calculating the critical exponents of the Ising model as a conformal fermion theory. For two dimensions, it is convenient to work in complex coordinates z and $\bar{z}$

Consider the two point functions for the spin operator σ and the energy density operator ϵ [5]

$$\langle \sigma_0 \sigma_r \rangle = \frac{1}{|r|^{d-2+\eta}} \tag{10}$$

$$\langle \epsilon_0 \epsilon_r \rangle = \frac{1}{|r|^{d/2-2/\nu}} \tag{11}$$

In a conformally invariant theory, scaling the axes as $x \to \lambda x$ will also scale the operators as $\sigma \to \lambda^{-\Delta_\sigma}\sigma$ and $\epsilon \to \lambda^{-\Delta_\epsilon}\epsilon$, where $\Delta_{\mathcal{O}}$ is the scaling dimension of an operator $\mathcal{O}$. In the case of the 2D free fermion theory, the energy density is proportional to the mass term ($\epsilon \propto \Psi\bar{\Psi}$). Thus, a simple dimensional analysis reveals that $\Delta_\epsilon = 1$. [5]

The spin operator σ is nonlocal in the fermion theory [5], and thus requires analysis of operator product expansions to directly calculate its scaling dimension. A simpler method is to calculate the central charge of the conformal theory by calculating the two point function of the energy momentum tensor T.

Looking at the action (6), we can easily read off the two point functions for the fermion operators

$$\langle \Psi(z_1)\Psi(z_2) \rangle = \frac{1}{z_1 - z_2} \tag{12}$$

$$\langle \bar{\Psi}(\bar{z}_1)\bar{\Psi}(\bar{z}_2) \rangle = \frac{1}{\bar{z}_1 - \bar{z}_2} \tag{13}$$

By Nöether's theorem, $T = -\frac{1}{2}\Psi\partial\Psi$. Therefore, Wick's Theorem give us [8]

$$\begin{aligned} \langle T(z_1)T(z_2) \rangle &= \frac{1}{4}\Big[\langle \Psi(z_1)\partial\Psi(z_2)\rangle \langle \partial\Psi(z_1)\Psi(z_2)\rangle - \langle \Psi(z_1)\Psi(z_2)\rangle \langle \partial\Psi(z_1)\partial\Psi(z_2)\rangle \Big] \\ &= \frac{1}{4(z_1-z_2)^4} \\ &= \frac{c}{2(z_1-z_2)^4} \end{aligned}$$

which tells us the central charge $c = \frac{1}{2}$

From knowledge of the central charge and the scaling dimension of ϵ, we can identify the free fermion theory as the minimal unitary model $\mathcal{M}_{(4,3)}$, with $\Delta_\sigma = \frac{1}{8}$. Checking (10) and (11), we indeed get the correct critical exponents $\eta = \frac{1}{4}$ and $\nu = 1$ for the 2D Ising model (which we can calculate directly from the Onsager solution).

VII. THE 3D ISING MODEL AS A FERMIONIC STRING THEORY

A similar analysis potentially allows us to probe the critical exponents for the 3D Ising model. Dotsenko shows that the 3D Ising model can be represented as a fermionic string theory. Consider placing four fermions $\psi_\mu^\alpha(x)$ on every link of the lattice. The corresponding fermionic theory is given by the partition function [4]

$$Z = \int \mathrm{D}\psi \, \exp\Big\{ \sum_{links} \bar{\psi}(x)\psi(x) \\ + t \sum_{plaqs} \psi_\mu^\alpha(x) \bar{\psi}_\nu^\delta(x+\hat{\mu}) \bar{\psi}^\beta \mu(x+\hat{\nu}) \psi_\nu^\gamma(x) \hat{M}_{\mu\nu}^{\alpha\beta,\gamma\delta} \quad (14)$$

where $\bar{\psi} = C^{-1}\psi$, $t = \tanh(\beta J_i)$. C and $\hat{M}$ are not unique; one such choice for C and $\hat{M}$ are given explicitly in [9]. For shorthand, we refer to the plaquette term as $(\psi\bar{\psi}\bar{\psi}\psi)$.

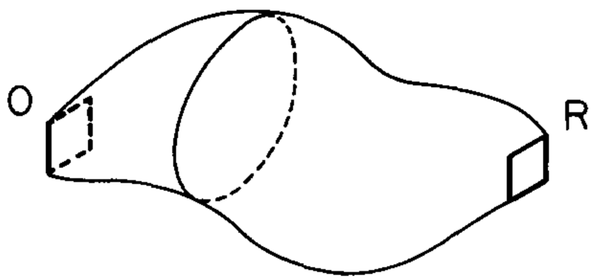

FIG. 1. A surface connecting two plaquettes

The energy density operator in this case is simply $\epsilon_{x,\mu\nu} = (\psi\bar{\psi}\bar{\psi}\psi)_{x,\mu\nu}$, [4] which generates a plaquette on the lattice. Integrating over the fermionic fields, $\langle\epsilon_0\epsilon_r\rangle$ is equal to a sum over surfaces whose boundaries are the two plaquettes at $x = 0$ and $x = r$ (see Figure 1), each surface contributing a factor $\sim t^{A/a^2}$, where A is the surface area and a is the lattice spacing.

In the continuum limit, $\langle\epsilon_0\epsilon_r\rangle$ is hypothesized to become a sum over worldsheets between two string loops. At the critical point, these strings become massless (analogous to the 2D Ising model). However, a major complication to calculating Δ_ϵ is the fermionic nature of the strings, requiring us to take special care when a surface crosses over itself. [4]

If a surface S has lines of self-intersection with total length l, an additional sign factor of $\Phi[S] = (-1)^{l/a}$ must be included. In the continuum limit, Kavalov and Sedrakyan showed that the sign factor induces a free 2-component dirac theory defined on the surface, with a string theory partition function

$$Z = \sum_{\partial S=0} \int \mathrm{D}\bar{\psi}\mathrm{D}\psi \, \exp\Big\{ \frac{A}{a^2} \\ + \frac{i}{2}\int \mathrm{d}^2\xi \, \sqrt{g}\big[\bar{\psi}\gamma^\alpha\partial_\alpha(\psi) - \partial_\alpha(\bar{\psi})\gamma^\alpha\psi\big]\Big\} \quad (15)$$

where $\sqrt{g}$ is the square root of the determinant of the induced metric $g_{\alpha\beta}(\xi) = \vec{x}_\alpha(\xi)\cdot\vec{x}_\beta(\xi)$ of the worldsheet, $\gamma^\alpha = \vec{x}_\alpha(\xi)\cdot\vec{\sigma}$ are local (2x2) gamma matrices, and $\vec{x}_\alpha(\xi)$ are the basis vectors tangent to the surface at ξ.

VIII. MODERN STUDIES OF THE 3D ISING MODEL

The fermionic string representation of the 3D Ising model at the moment is unsolved. Additionally, it has yet to be fully proved that the continuum limit is indeed a string theory. One major problem is that a string theory has significantly more degrees of freedom and particle spectra than the Ising model permits. However, Distler argues that these unwanted particles would be removed due to the additional sign factor of the fermionic strings. [10]

Current numerical methods for calculating the critical exponents of the 3D Ising model involve looking at the conformal field theory of the spin operators directly, instead of working with fermionic operators. [11] These conformal bootstrap methods have been able to compute the critical exponents of the 3D Ising model to a greater precision than any other method to date.

[1] L. Onsager, Phys. Rev. **65**, 117 (1944).
[2] F. A. Berezin, Russian Mathematical Surveys **24**, 1 (1969).
[3] V. N. Plechko, Journal of Physical Studies **1**, 554 (1997), cond-mat/9812434.
[4] V. Dotsenko, Nuclear Physics B **285**, 45 (1987).
[5] P. Di Francesco, P. Mathieu, and D. Sénéchal, *Conformal Field Theory*, Graduate texts in contemporary physics (Island Press, 1996).
[6] C. Itzykson and J.-M. Drouffe, *Statistical Field Theory*, Cambridge Monographs on Mathematical Physics, Vol. 1 (Cambridge University Press, 1989).
[7] V. N. Plechko, Physica A: Statistical Mechanics and its Applications **152**, 51 (1988).
[8] C. Itzykson and J.-M. Drouffe, *Statistical Field Theory*, Cambridge Monographs on Mathematical Physics, Vol. 2 (Cambridge University Press, 1989).
[9] V. Dotsenko and A. Polyakov, Advanced Studies in Pure Mathematics , 171 (1988).
[10] J. J. Distler (1992).
[11] S. El-Showk, M. F. Paulos, D. Poland, S. Rychkov, D. Simmons-Duffin, and A. Vichi, Journal of Statistical Physics **157**, 869 (2014).