Field Theory and Quantum Fluctuations of Fluid Membranes

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We review effective field theories of fluid membranes, including the role of topology, and outline two kinds of phase transitions due to quantum fluctuations. We second-quantize the quantum fluctuations of idealized fluid membranes at low temperature. We illustrate how to find the spectrum of fluctuations for the simple examples of a sphere and a torus by introducing isothermal coordinates.

I. INTRODUCTION

Cell membranes are made of amphiphilic lipid molecules - molecules which have both hydrophilic and hydrophobic parts. These lipids form a certain self-assembled structure which is thermodynamically stable; for example, many cells in humans are made of a lipid *bilayer*, which is mobile and deformable. In the limit that the variations in shape of the cell membrane are much larger than the size of the lipids themselves, we may coarse-grain the cell membrane and treat it as a smooth surface, or 2-manifold. This invites the use of geometry and topology to attack physical problems. On the basis of symmetry and invariance arguments, we first construct the continuum field theory describing the 2-manifold, which was first obtained by Helfrich [1]. In particular, a 2-manifold is of the same dimension as the world-sheet swept out by a string, and the continuum model of membrane energy was first derived in the context of string theory by Polyakov [2].

Although real cell membranes would freeze long before quantum fluctuations become important, this paper is concerned with *quantum*, rather than thermal, fluctuations. We then review the implications of Helfrich theory on two different kinds of phase transitions for low temperatures, driven by quantum fluctuations [3, 4], and also the appearance of topological quantities in the partition function of a quadratic Hamiltonian on a curved membrane [5]. Finally, following the canonical quantization of a string [6], we second-quantize the quantum fluctuations of a membrane for a quadratic Hamiltonian. In simple geometries, we can both obtain the energy spectrum and visualize what these fluctuations look like by introducing isothermal coordinates [7].

II. NOTATION AND MATHEMATICAL RESULTS

Consider a membrane (2-manifold) in d-dimensional space, and let $\vec{X} \in \mathbb{R}^d$ describe points in this d-space. The surface is 2-dimensional, so we use $\vec{\sigma} = (\sigma^1, \sigma^2)$ as local coordinates on the manifold. The *metric tensor*

$$g_{ij}(\vec{\sigma}) := \frac{\partial \vec{X}}{\partial \sigma^i} \frac{\partial \vec{X}}{\partial \sigma^j}$$

gives the Euclidean distance between two point on the function $\vec{X}(\vec{\sigma})$.

The area form is $d^2A = \sqrt{g} d\sigma^i \wedge d\sigma^j = \sqrt{g} d\sigma^i d\sigma^j$, where $g = \det g_{ij}$.

Let $R_{1,2}$ be the principal radii of curvature, which can be locally defined, for example, via the *extrinsic curvature tensor* $\vec{K}_{ij}$ [7]. The *Gaussian curvature* K and the *mean curvature* H are defined in $d = 3$ as

$$K = \frac{1}{R_1 R_2} \text{ and } H = \frac{1}{2}(R_1^{-1} + R_2^{-1}).$$

A useful formula for H, which we will use later, is

$$H = \frac{1}{2} g^{ij} \vec{K}_{ij}, \text{ where } \vec{K}_{ij} = D_i D_j \vec{X}.$$

For $d = 3$, there is only one direction for $\vec{K}$, so $\vec{K}$ is in fact a scalar.

Besides these definitions, we will also use a few theorems from differential geometry. Gauss' *theorem egregium* says that the Gaussian extrinsic curvature K is related to the Ricci curvature R,

$$2K = R.$$

The *Gauss-Bonnet theorem* says that, for a closed manifold without boundary (such as most cells),

$$\int d^2A \sqrt{g} R = 4\pi\chi = 8\pi(1 - g),$$

where χ is the Euler characteristic, g is the genus, and R is the Ricci curvature.

The *Laplace-Beltrami operator* for functions on a manifold is defined to preserve an "integration by parts" for a scalar function f:

$$-\langle \text{div } X, f \rangle = \langle X, \nabla f \rangle,$$

where ∇ is the covariant derivative. The explicit form is

$$\Delta f = \frac{1}{\sqrt{g}} \partial_i (g^{ij} \sqrt{g} \partial_j f).$$

In practice, this means that integration by parts, for example on a scalar field ϕ, looks like

$$\int d^2\sigma (\nabla\phi)^2 = -\int d^2\sigma \phi \Delta\phi.$$

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III. EFFECTIVE FIELD THEORY OF FLUID MEMBRANES

These effective theories of fluid membranes assume that the characteristic time for in-plane deformations is much less than the time it takes for the membrane shape itself to change, so effectively, the energy can be described only in terms of the membrane shape. This resembles the Born-Oppenheimer approximation of quantum chemistry, where we care only about the positions of the atomic nuclei and not about the electrons themselves.

A. Construction of Helfrich energy

The **Helfrich energy**, which was derived on phenomenological considerations by Helfrich [1], is

$$\mathcal{H} = \int d^2A(\frac{\kappa}{2}H(\vec{\sigma})^2 + \bar{\kappa}K(\vec{\sigma})).$$

We can construct this by arguing that $\mathcal{H}$ must be only a function of $\vec{X}(\vec{\sigma})$ and that it must be translationally and rotationally invariant, and reparametrization invariant. We keep only those terms which are relevant in the IR. In other words,

- Rotational and translational invariance: $\mathcal{H}$ must be invariant under $X' = \mathcal{R}X + a$ and $g' = g$, where $\mathcal{R}$ is a rotation matrix.
- Diffeomorphic (reparametrization) invariance: $\mathcal{H}$ must be invariant under $X' = X$ and $g'_{ij}\frac{\partial\sigma'^i}{\partial\sigma^k}\frac{\partial\sigma'^j}{\partial\sigma^l} = g_{kl}$, for new coordinates σ' and old coordinates σ.
- Weyl (conformal) invariance: $\mathcal{H}$ must be invariant under $X' = X$, $g' = e^{\phi(\sigma)}g$.

Polchinski [6] shows that for a manifold without boundary,

$$\int_M d^2\sigma\sqrt{g}R,$$

where R is the scalar curvature (or Ricci scalar), is invariant under all three operations above. Under a Weyl rescaling,

$$\sqrt{g'}R' = \sqrt{g}(R - 2\Delta\phi),$$

and $\sqrt{g}\nabla_i v^i = \partial_i(-\sqrt{g}v^i)$ for any vector i, so the integral of the variation vanishes for a manifold without boundary. As in Homework 1, Problem 4 of this course [8], the derivatives which are invariant under translations and rotations are

$$\partial_i\vec{X}_j \cdot \partial_i\vec{X}_j \text{ and } \partial_{ii}\vec{X}_j \cdot \partial_{kk}\vec{X}_j,$$

which, when generalized to manifolds M with curvature and integrated over the surface , give the contributions (up to multiplicative constants)

$$\int dA \text{ and } \int dA(\Delta\vec{X})^2,$$

and we do not care about higher derivatives because they will be irrelevant in the IR. Putting these three terms together gives the general Hamiltonian

$$\mathcal{H} = r_0\int d^2\sigma\sqrt{g} + \frac{\kappa_0}{2}\int d^2\sigma\sqrt{g}(\Delta\vec{X})^2 + \frac{\bar{\kappa}_0}{2}\int d^2\sigma\sqrt{g}R.$$

The $(\Delta\vec{X})$ term can be interpreted as $(\frac{1}{2}g^{ij}D_iD_j\vec{X})$, which is merely the mean curvature, H. Thus, this term returns the H^2 term in the Helfrich energy. The $\int d^2\sigma\sqrt{g}R$ term returns the other term in the Helfrich energy via Gauss' *theorem egregium*, $R = 2K$.

Here, r_0 is interpreted as the **surface tension**, which turns out to be relevant, and $\kappa_0, \bar{\kappa}_0$ as the bending rigidity and the Gaussian rigidity, respectively, which both turn out to be marginal. You can guess this behavior by rescaling $\vec{X} \to \lambda\vec{X} \iff q \to \lambda^{-1}q$, where $\lambda > 1$: $r_0 \to \lambda^2 r_0$ and the curvature parameters are unchanged. Because of the Gauss-Bonnet theorem, the $\int dAR$ contribution can be ignored unless the Euler characteristic χ of the manifold changes.

In fact, Helfrich introduced a **spontaneous curvature** H_0 to describe the shape of certain membranes, like those of red blood cells. The energy is modified to

$$\mathcal{H} = \int d^2A(\frac{\kappa}{2}(H(\vec{\sigma}) - H_0)^2 + \bar{\kappa}K(\vec{\sigma})).$$

Variation of this energy returns the *shape equation* for the mean curvature $H(\vec{\sigma})$. Tuning the numerical parameters can produce a solution of the shape equation which resembles the collapsed-sphere shape of a red blood cell [1, 9]. This is a mean-field theory; because I am focusing on fluctuations, I will not describe it here.

B. Phase transitions due to quantum fluctuations

In this section, we review two kinds of crumpling transitions, both of which are due to quantum fluctuations. The first is a transition with respect to *temperature*: there exists a critical temperature T_c such that the membrane is flat for $T < T_c$ and crumpled for $T > T_c$[3]. The second is a transition with respect to the size of Planck's constant, $\hbar$: there exists a critical value $\hbar^*$ such that the membrane is flat for $\hbar < \hbar^*$ and crumpled for $\hbar > \hbar^*$[4]. Although it is not physically relevant, it is interesting because such a phase transition, being a true quantum phase transition, can occur even at $T = 0$.

Both treatments make use of the *Monge representation*, in which the two-dimensional coordinates $\vec{\sigma} = (\sigma_1, \sigma_2)$ are located on a flat plane lying underneath the membrane, rather than being local coordinates on the membrane itself. The position in $d = 3$ of a point on a two-dimensional membrane is now described as

$$\vec{X}(\vec{\sigma}, t) = (\vec{\sigma}, f(\vec{\sigma}, t)).$$

Assuming only continuous deformations, we may ignore the topological term $\int d^2\sigma\sqrt{g}R$ in the Helfrich energy,

add a kinetic term, and start with

$$\mathcal{H} = \int d^2\sigma\sqrt{g}(r + \frac{\lambda}{2}(\partial_t \vec{X})^2 + \frac{2}{\lambda}H^2).$$

The relation between coefficient on the kinetic term and H term have been chosen for convenience. The quantum action $\frac{S}{\hbar}$ is expanded to fourth order in the "height" variable f as

$$\frac{1}{\hbar}\int_{t,\vec{\sigma}}\left(\frac{\lambda}{2}(\partial_t f)^2+\frac{r}{2}(\partial_i f)^2+\frac{1}{2\lambda}(\partial^2 f)^2-\frac{\lambda}{4}(\partial_t f)^2(\partial_i f)^2-\frac{r}{8}(\partial_i f)^2(\partial_j f)^2-\frac{1}{4\lambda}(\partial_i f)^2(\partial^2 f)^2-\frac{1}{\lambda}(\partial_i f\partial_j f)(\partial_i\partial_j f)(\partial^2 f)\right)$$

The crumpling transition is equivalent to asking the question: does the rigidity λ flow to 0 or ∞ in the IR? If $\lambda \to 0$, $\frac{1}{\lambda} \to \infty$, so the membrane is "stiff" and hence flat. If $\lambda \to \infty$, the membrane is "floppy" and hence crumpled.

Crumpling at high temperature: We use the $\mathcal{O}(f^4)$ terms to renormalize the parameters in the quadratic part of the energy,

$$\frac{S_0}{\hbar} = \int dt d^2\sigma\left(\frac{\lambda}{2\hbar}(\partial_t f)^2 + \frac{r}{2\hbar}(\partial_i f)^2 + \frac{1}{2\hbar\lambda}(\partial^2 f)^2\right).$$

As outlined in Borelli [3], the result of Wilsonian perturbation theory is the flow equation

$$\beta(\lambda) = \mu\partial_\mu\lambda = (1 - d - \frac{3u_{\text{qm}} - 4u_{\text{th}}}{16\pi})\lambda.$$

$u_{\text{qm}} = \hbar r_0\lambda_0, u_{\text{th}} = kT\lambda_0$ are suitably chosen expansion parameters in terms of the unrenormalized operators r_0, λ_0. There is a corresponding flow equation for r. The main takeaway is that there exists a nontrivial fixed point

$$(r = 0, \lambda = \lambda^*(T))$$

such that the flow of λ away from this fixed point changes direction at $T = T^*$. For $T < T^*$, λ flows to 0; for $T > T^*$, λ flows to ∞. This can be interpreted as a competition between the expansion parameters u_{qm} and u_{th}. When thermal fluctuations are large enough (i.e. for large enough T), the effect of u_{th} (thermal fluctuations) overpowers that of u_{qm} (quantum fluctuations).

Crumpling at high $\hbar$: What if we allow $\hbar$ to change? As outlined in Foltin [4], the flow equation for λ is

$$\beta(\lambda) = -\frac{\beta(\hbar)}{\epsilon}\lambda,$$

where $\beta(\hbar) = \hbar(d - \hbar)$. This means that as $\mu \to 0 \implies \lambda \to 0$ for $\hbar < \hbar^* = d$, and $\mu \to 0 \implies \lambda \to \infty$ for $\hbar > \hbar^* = d$. There is thus a crumpling transition at $\hbar = \hbar^* = d$. The membrane is flat for $\hbar < \hbar^*$ and crumpled for $\hbar > \hbar^*$.

C. Free field on a surface

Our aim here is to compute the most divergent terms in the partition function Z of a massless field on a curved surface. We will take the leading (quadratic) terms from the $\mathcal{O}(f^4)$ expansion of the Helfrich energy in the previous subsection, to avoiding performing perturbation theory but still return interesting mathematical results. The result is that the coefficients of the partition function turn out to be related to the topological properties of the surface [5].

Go back to the parametrization where $\vec{\sigma} = (\sigma_1, \sigma_2)$ are local coordinates of points on the surface, rather than Monge coordinates. Consider a massless free field $\phi(\sigma)$ on a 2-manifold with action

$$S = \frac{1}{2}\int_M d^2\sigma\sqrt{g}g^{ij}(\partial_i\phi)(\partial_j\phi).$$

If M has no boundary, then

$$S = \frac{1}{2}\int_M d^2\sigma\sqrt{g}\phi(-\Delta)\phi$$

$$\implies Z = \int D\phi e^{-S[\phi]} \sim \frac{1}{\sqrt{\det(-\Delta)}}.$$

However, defining the measure $D\phi$ is nontrivial; we require it to be reparameterization invariant. The simplest way to do so is to write

$$\phi = \sum_{k=0}^{\infty} a_k\phi_k,$$

where

$$\Delta\phi_k = -\lambda_k\phi_k, \langle\phi_k|\phi_l\rangle = \int_M d^2\sigma\sqrt{g}\phi_k\phi_l = \delta_{kl}.$$

We see that k is like a momentum and $\lambda_k \sim k^2 > 0$ is like an energy. Then

$$D\phi = \prod_k da_k \text{ and } \det(-\Delta) = e^{\sum_k \ln\lambda_k}.$$

There are two issues that arise:

1. $\lambda_0 = 0$: This makes the determinant equal to zero. We will just ignore the $k = 0$ mode; this is equivalent to assuming that the manifold (i.e. red blood cell) has a finite size.

2. As $k \to \infty$, $\lambda_k \sim k^2$ gives a divergent contribution. We will introduce a UV cutoff in a reparameterization-invariant way. (However, the divergent contributions might still have regulator-dependence.) Because

$$\det(-\nabla^2) \sim e^{\sum_k \ln \lambda_k} \sim e^{\Lambda^2},$$

we expect

$$\ln Z = B\Lambda^2 + C\ln(\Lambda^2) + D + \mathcal{O}(\Lambda^{-2}).$$

We will compute these coefficients in terms of the geometry of the surface.

Heat-kernel regularization: Let a prime, $'$, omit the $k = 0$ mode. This method will utilize the Green function of the Laplace, or heat, equation. Let $\epsilon = \frac{1}{\Lambda^2}$ be a cutoff and note that

$$\log \det{}'(-\Delta\epsilon) = \mathrm{tr}'[\log(-\Delta\epsilon)]$$

$$= -\mathrm{tr}'[\int_\epsilon^\infty \frac{dt}{t} e^{t\Delta}] = -\int_\epsilon^\infty \frac{dt}{t}\mathrm{tr}'(e^{t\Delta}).$$

This follows by noting that

$$-\int_\epsilon^\infty \frac{dt}{t} e^{-\lambda_k t} = -\int_{\lambda_k\epsilon}^\infty \frac{du}{u} e^{-u} \sim \log(\lambda_k \Delta),$$

at least when ϵ is small, so $e^{-u} \approx 1$.

Now, we use the fact that

$$\mathrm{tr}'(e^{t\Delta}) = (\int_M dA G(\sigma', \sigma; t)) - 1.$$

Note that $\mathrm{tr}'(e^{t\Delta})$ clearly decreases with time. Here, $G(\sigma, \sigma'; t)$ is the solution to the differential equation $\partial_t G = \nabla_\sigma^2 G$ with initial condition $G(\sigma, \sigma'; t = 0) = \frac{\delta^{(2)}(\sigma - \sigma')}{\sqrt{g}}$.

This follows by assuming the general form

$$G(\sigma', \sigma; t) = \sum_{k=0}^\infty a_k \phi_k(\sigma) e^{-\lambda_k t}$$

and matching to the initial conditions. We find $a_k = \phi_k(\sigma')$ and therefore

$$G(\sigma', \sigma; t) = \sum_{k=0}^\infty \phi_k(\sigma')\phi_k(\sigma) e^{-\lambda_k t}$$

$$\approx \sum_{k=0}^\infty \phi_k(\sigma)\phi_k(\sigma) e^{-\lambda_k t},$$

if we assume that $\phi(\sigma') \approx \phi(\sigma)$, this gives

$$\mathrm{tr}'(e^{t\Delta}) = \int_M d^2\sigma\sqrt{g} G(\sigma', \sigma; t).$$

This is a reasonable approximation to make when ϵ is the important parameter and is very small. Because the divergence is for $t \to \epsilon$, small values of t mean that the diffusive wave has not spread out very much, which means $\sigma' \approx \sigma$ is a fine approximation to make.

For short times,

$$G(\sigma, \sigma'; t) \approx \frac{1}{4\pi}(\frac{1}{t} + \frac{R(\sigma)}{6} + \mathcal{O}(t)),$$

where R is the Ricci curvature at point σ [5]. This, of course, looks a bit like the solution to the diffusion equation, $\sim e^{-x^2/t}$. Immediately (putting back $\epsilon = \Lambda^{-2}$) integration over t gives

$$\ln Z = \frac{-\Lambda^2}{4\pi}\int_M d^2\sigma\sqrt{g} + \frac{\log\Lambda^2}{24\pi}\int_M d^2\sigma\sqrt{g}R + \mathcal{O}(1).$$

Conclusion:

$$B \propto \text{surface area}, C \propto \text{Euler characteristic}, \chi.$$

IV. SECOND QUANTIZATION OF OSCILLATIONS OF A CLOSED 2-MANIFOLD

It is possible to use the formalism developed above and second-quantize the quantum fluctuations of a fluid membrane, following Polchinski's quantization for open strings [6]. Although this is irrelevant for biological membranes, because they would freeze long before temperatures became low enough for quantum fluctuations to be relevant, this treatment is relevant for non-biological membranes, such as the interfaces between different phases in superfluid helium; for example, see Marchenkov et. al [10].

Assume we have a 2-manifold parametrized by atlas coordinates (s_1, s_2) which lives in 3-space, described by $\vec{X} = (X_1, X_2, X_3)$. Because $3-2 = 1$, locally there is only one direction orthogonal to the surface of the manifold. The small displacement of the membrane can be treated as orthogonal to its surface,

$$\delta\vec{X} \perp \hat{s}_1, \hat{s}_2,$$

so we can simply treat this displacement $\delta\vec{X}$ as a scalar field ϕ (i.e. because it is one-dimensional). Hence, the machinery developed in the previous section is still applicable.

Because I do not expect there to be a gap in the spectrum, I will not add a mass term to the action. I will however, add a kinetic term:

$$S = \int d\tau d^2 s\sqrt{g(s)}(g^{ij}\partial_i\phi\partial_j\phi - (\partial_\tau\phi)^2).$$

As in the previous section, the equation of motion can be derived by variation of the action with respect to ϕ and integration by parts. The result is

$$(\Delta - \partial_\tau^2)\phi(s) = 0,$$

where Δ is the Laplace-Beltrami operator. This gives an approximate behavior

$$\phi_k(\tau) \sim e^{\pm\lambda_k \tau}$$

for the kth oscillatory mode. The eigenfunction is described by the eigenvalue λ_k. The general time-dependent solution for the position of a particular point on the membrane is

$$\vec{X}(\vec{s},\tau) = \vec{X}(\vec{s},0) + \vec{v}\tau + (\hat{s}_1 \times \hat{s}_2) \sum_{k=-\infty}^{\infty} \alpha_k \phi_k(\vec{s},\tau).$$

In the above, $\vec{X}(\vec{s},0)$ is the starting point (in 3-space) of the point on the membrane with atlas coordinates $\vec{s}$, $\vec{v}$ is an overall drift velocity of the entire membrane, $\hat{s}_1 \times \hat{s}_2$ is the fluctuation direction (locally orthogonal to the surface), and the sum runs over all fluctuations. I implicitly assumed the membrane was originally in a nondegenerate ground state, so there are no trends in the overall shape, other than fluctuations.

An individual quantum mode is described by $\phi_k(\vec{s},\tau)$, with appropriate normalization and boundary conditions. In real-time, the eigenfunctions go as $e^{-i\omega t}$, so we have

$$\frac{1}{T}\int_0^T dt e^{-i(\lambda_k+\lambda_l)t} = \delta_{k,-l},$$

$$\int d^2s\sqrt{g}\phi_k(\vec{s})\phi_l(\vec{s}) = \delta_{k,\pm l}.$$

We would like to introduce creation and annihilation operators. This is possible by introducing canonical commutators,

$$[\phi(\vec{s},t), \pi(\vec{s}',t')] = i\delta(\vec{s}-\vec{s}')\delta(t-t'),$$

where $\pi(\vec{s},t) = \partial_t\phi(\vec{s},t)$ is the canonically conjugate momentum.

Introduce the real-time conventions

$$\phi(\vec{s},t) = \sum_{k=-\infty,k\neq 0}^{\infty} \frac{\alpha_k}{\sqrt{\lambda_k}}\phi_k(\vec{s})e^{-i\lambda_k t},$$

$$\pi(\vec{s},t) = \sum_{l=-\infty,l\neq 0}^{\infty} -i\alpha_l\sqrt{\lambda_l}\phi_l(\vec{s})e^{-i\lambda_l t}.$$

Because $\phi,\pi \in \mathbb{R}$, $\lambda_{-k} = -\lambda_k, \alpha_{-k} = \alpha_k^*$, and $\phi_{-k}(\vec{s}) = \phi_k(\vec{s})$. Above, $\sqrt{\lambda_k}$ is allowed to be imaginary. We find that

$$[\phi(\vec{s},t),\pi(\vec{s}',t')] = i\sum_{kl}[\alpha_k,\alpha_l]\phi_k(\vec{s})\phi_l(\vec{s}')e^{-i(\lambda_k+\lambda_l)t}.$$

Integrating over $\int d^2s$ and $\frac{1}{T}\int dt$ and using the normalizations above gives the commutations

$$[\alpha_k,\alpha_l] = \delta_{k,-l}.$$

If we constrain $k \geq 0$ and write $\alpha_{-k} = \alpha_k^\dagger$, we have

$$[\alpha_k,\alpha_l^\dagger] = \delta_{k,l}$$

which is the usual commutation. Essentially, the scalar relation $\alpha_{-k} = \alpha_k^*$ is promoted to the operator relation $\alpha_{-k} = \alpha_k^\dagger$. Hence, we can write the spectrum in terms of a "ladder" of vibrational occupations, for each positive energy state λ_k. The general excited state $|\Psi\rangle$ can be written

$$|\Psi\rangle = \prod_{k=1}^{\infty}\frac{(\alpha_k^\dagger)^{n_k}}{\sqrt{n_k!}}|0\rangle.$$

If the membrane lives in $d > 3$ dimensions and we diagonalize the oscillations into uncoupled directions $3 \leq i \leq d$, the general excited state is

$$|\Psi\rangle = \prod_{i=3}^{d}\prod_{k_i=1}^{\infty}\frac{(\alpha_{k_i}^\dagger)^{n_{k_i}}}{\sqrt{n_{k_i}!}}|0\rangle.$$

V. SPECTRUM VIA CONFORMAL COORDINATES

We saw above that we can interpret the quantum fluctuations of a fluid membrane as a collection of bosonic phonon modes, each with a different energy. The occupation of these modes is governed by the Bose-Einstein distribution, so determining the spectrum can yield, for example, the energy $E(T)$ and hence specific heat $C(T)$ of the fluid membrane. The main issue with the above quantization is the difficulty of diagonalizing the Laplace-Beltrami operator, Δ. Unlike in flat space, the Laplace operator is generally not diagonal in the different components, essentially because the metric is no longer diagonal.

There are several ways to solve this. One method is to use computational programs, such as [11]. For simple surfaces, we can diagonalize the metric and hence also diagonalize Δ through the technique of introducing *conformal coordinates* or *isothermal coordinates*, which have induced metric [7]

$$ds^2 = g(u,v)(du^2+dv^2).$$

The Laplace operator takes a very simple form,

$$\Delta = \partial_u^2 + \partial_v^2.$$

It can be proved that such a diagonalization always exists, though it can be found analytically only for simple cases. Let us consider two examples.

A. Sphere

The conformal coordinates of a sphere of radius R can be found by stereographic projection onto the xy-plane.

The result is

$$ds^2 = \frac{4R^2}{(R^2+x^2+y^2)}(dx^2+dy^2).$$

The xy-plane in question extends to infinity, $-\infty \leq x, y \leq \infty$, so we switch to (r,θ) polar coordinates. According to Kuttler [12], the eigenfunctions and eigenvalues of the Laplace operator for a circular membrane are

$$u_{mn} = \mathcal{J}_m(\frac{j_{mn}r}{a})(A\cos(m\theta)+B\sin(m\theta)),$$

$$\lambda_{mn} = (\frac{j_{mn}}{a})^2, m \in \mathbb{Z}^0, n \in \mathbb{Z}^+,$$

where j_{mn} is the nth zero of the mth Bessel function. In this stereographic projection, the eigenfunction $\phi_k(\vec{s})$ should be single-valued at $r \to \infty$. Because the Bessel function tends to zero, $J_m(x) \to 0$ as $x \to \infty$, this is guaranteed anyway. So, the above roots j_{mn} give the correct spectrum. Although $u_{mn} \to 0$ for $r \to \infty$, it is not necessarily true that $\phi_k(\vec{s}) \to 0$ as $r \to \infty$. This is because we can add a solution η of the homogeneous equation $\Delta\eta(\vec{s}) = 0$ to $\phi_k(\vec{s})$ and preserve the eigenvalue.

B. Torus

The conformal coordinates of a torus, originally described by $(\sqrt{x^2+y^2}-R_1)^2+z^2 = R_2^2$, can be expressed in the dimensionless "aspect ratio" $r := \frac{R_1}{R_2}$ as[13]

$$ds^2 = R_2^2(\frac{r^2-1}{r-\cos\phi})^2(d\theta^2+\frac{d\phi^2}{r^2-1}) = g(\theta,\tilde{\phi})(d\theta^2+d\tilde{\phi}^2).$$

Here, θ is the usual polar angle in two-dimensional (r,θ) coordinates. $\tilde{\phi} = (r^2-1)^{-1/2}\phi$, and ϕ is expressed in terms of α, which describes vertical position (for example, $z = R_2\sin\alpha$ in the Cartesian representation).

$$\cos\alpha = \frac{r\cos\phi - 1}{r-\cos\phi}.$$

Because $0 \leq \theta, \phi \leq 2\pi$, the torus is conformally equivalent to a rectangle. This is aesthetically pleasing because we know that we can fold up a rectangle to make a torus, by first making a cylinder and then connecting the ends of the cylinder. The eigenfunctions and eigenvalues of the Laplace equation for a rectangular region are [12]

$$u_{mn}(\theta,\tilde{\phi}) = \sin(\frac{m\pi\theta}{a})\sin(\frac{n\pi\tilde{\phi}}{b}),$$

$$\lambda_{mn} = \pi^2((\frac{m}{a})^2+(\frac{n}{b})^2).$$

Here, $m,n \in \mathbb{Z}^+$, $a = 2\pi, b = \frac{2\pi}{\sqrt{r^2-1}}$.

C. Computation of physical quantities

For agreement with the eigenvalues λ_{mn} above, let us label eigenmodes with (mn) instead of with k. The average number of phonons in a given mode is given by the Bose-Einstein distribution,

$$\langle N_{mn}\rangle = \frac{1}{e^{\beta(\lambda_{mn}-\mu)}-1}.$$

The partition function is merely

$$Z = \prod_{mn}\Big(\sum_{N_{mn}=0}^{\infty} e^{-\beta N_{mn}\lambda_{mn}}\Big) = \prod_{mn}\frac{1}{1-e^{-\beta\lambda_{mn}}}.$$

and the energy and specific heat $E(T), C(T)$ follows from differentiating with respect to $\beta = \frac{1}{kT}$. Thus, if we are interested only in the thermal behavior of a membrane, the second quantization gives a much simpler partition function than the typical partition function of field theory, $Z = \int \mathcal{D}\phi e^{-\beta\mathcal{H}[\phi]}$.

VI. CONCLUSION

In this paper, we derived the coarsed-grained (Helfrich) model of a fluid membrane via invariance arguments and studied its quantum fluctuations. In particular, we found a crumpling transition temperature T^* at which thermal fluctuations overpower quantum fluctuations, and a critical value $\hbar^*$ at which there is a crumpling transition even at $T = 0$. We found that topological quantities (surface area, Euler characteristic) appear in the computation of the divergent parts of the free energy, $\ln Z$. Finally, we studied individual quantum phonon modes by introducing creation and annihilation operators. Conformal coordinates help us visualize what these vibrational modes look like.

For the thermally-induced crumpling transition, it would be interesting to extend our results to account for boundary effects, because the systems for which this calculation is relevant, such as superfluid He, are manifolds with boundaries. For the quantization of thermal fluctuations, it would be interesting to try to build a quantum field theory out of fields such as

$$\phi(\vec{\sigma},t) = \sum_k \alpha_k\phi_k(\vec{s})e^{-i\lambda_k t} + \alpha_k^\dagger\phi_k(\vec{s})e^{i\lambda_k t}$$

and construct a Lagrangian out of these fields, which are now operators. This mirrors the procedure in relativistic field theory. It is conceivable that such a quantum field theory, which unlike Helfrich theory accounts for quantum fluctuations, could be generalized to systems more general than fluid membranes. For example, the curvatures H and K could also have relevance in surface science, where nonzero curvature is interpreted as a change in coordination number of an atom on the surface. The Lagrangian would then be expanded in terms

of the fields ϕ, and $\alpha_k^\dagger$ would be interpreted as creating a k-state phonon on the surface.

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