Defects in the $O(N)$ Model

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The critical phenomena associated with defects in the $O(N)$ model are discussed. We discuss codimension D defects in a d-dimensional $O(N)$ model using saddle point analysis and using the large N limit. The specific case of plane defects in $d = 4$ is described in some detail.

The surface critical behavior of $O(N)$ models has been studied both in field theory and in lattice models since the 1970s.[1–4] One may think of a surface as a co-dimension 1 defect. This point of view naturally leads to the question of *defect* critical behavior for co-dimension $D > 1$ defects. More concretely, one can ask what are the critical phenomena associated with $(d-D)$-dimensional scale-invariant manifolds of enhanced/suppressed couplings embedded in a d-dimensional $O(N)$ model. Such defects have been studied in some generality in the past.[5, 6] The discussion presented here is, in a sense, complementary to this line of work. We use a continuum field theory description analogous to Ref. [2] and Ref. [7] to describe the aforementioned defects. Using saddle point as well as large N approximation, we discuss $D \geq 1$ defects in analogy with surfaces. Lastly, we consider in some detail the specific case of plane defects in $d = 4$, which are particularly interesting since they correspond to defect lines in 3+1D quantum O(N) models.

The defects we consider in this paper will be described by equations of the form

$$x_{\perp}^{\mu} = 0 \tag{1}$$

where the index μ runs over the directions orthogonal to the defect manifold. In other words, the defect is a $d - D$ dimensional hyperplane with D directions orthogonal to it. Based on this, we define our Landau-Ginzburg energy functional as [2]

$$\beta\mathcal{H} = \int \mathrm{d}^d x \left[\sum_{i=1}^{N} \left(\frac{1}{2} t\phi_i^2 + \frac{1}{2} \left(\nabla\phi_i\right)^2 \right) + \frac{1}{4} u \left(\sum_{i=1}^{N} \phi_i^2 \right)^2 + \sum_{i=1}^{N} \left(\frac{1}{2} c\phi_i^2 - h_i\phi_i \right) \delta^{(D)}(x_{\perp}) \right] \tag{2}$$

where $\vec{\phi}(x) = (\phi_1(x), \phi_2(x), \cdots, \phi_N(x))$ is an N-component vector field. We have included a symmetry-breaking field localized on the defect which we will take to 0 by the end of all calculations. It serves as a helpful regulation scheme especially in the saddle point analysis.

I. SADDLE POINT ANALYSIS

The saddle point analysis amounts to solving for the field configuration that minimizes the Landau-Ginzburg energy functional. This results in the following equation:

$$-\nabla^2\phi_i + t\phi_i + u\phi_i \sum_{j=1}^{N} \phi_j^2 = (h_i - c\phi_i)\delta^{(D)}(x_{\perp}) \tag{3}$$

We will set the symmetry-breaking field h_i to point along the direction 1 in spin space, $h_i = h\delta_{i,1}$. Since $h_2 = h_3 = \cdots = h_N = 0$, we can set $\phi_2, \cdots, \phi_N$ to 0 too. Moreover, we can assume the saddle point solution to be independent of the coordinates that are parallel to the defect. Thus, we may express the saddle point field $\vec{\phi}$ as $\phi_i(x) = \phi(x_{\perp})\delta_{i,1}$. Using these simplifications, we can re-write Eq. 3 as

$$-\nabla_{\perp}^2\phi + t\phi + u\phi^3 = (h - c\phi)\delta^{(D)}(x_{\perp}) \tag{4}$$

Integrating Eq. 4 over a "cylinder" of vanishing radius ϵ surrounding the defect gives us the boundary condition:

$$-\lim_{\epsilon\to 0} S_D \epsilon^{D-1} \frac{\partial\phi}{\partial r_{\perp}}\Big|_{r_{\perp}=\epsilon} = h - c\phi(0) \tag{5}$$

where we used the divergence theorem to simplify the l.h.s. S_D is the solid angle in D dimensions. For $D = 1$, these equations reduce to

$$-\phi'' + t\phi + u\phi^3 = 0 \quad (x_{\perp} \neq 0) \tag{6}$$

$$\phi'(0^+) - \phi'(0^-) = c\phi(0) - h \tag{7}$$

where the primes denote derivatives w.r.t. $x_{\perp}$. Along with these, we have the boundary condition that ensures the field matches the bulk value at $x_{\perp} \to \pm\infty$. Eq. 6 can be integrated by multiplying both sides by $\phi'(x_{\perp})$,

$$\frac{1}{2}(\phi')^2 = \frac{t}{2}\phi^2 + \frac{u}{4}\phi^4 + k \tag{8}$$

where k is the constant of integration. For $t > 0$, $\phi(\pm\infty) = 0$ and $\phi'(x_\perp \to \pm\infty) = 0$. We can therefore set $k = 0$ in Eq. 8. Moreover, this boundary condition ensures that ϕ *decays* to 0 in the bulk, i.e. $\phi'(x_\perp)$ is negative (positive) for positive (negative) $x_\perp$. Using this in Eq. 7 gives us

$$\begin{aligned} h &= c\phi(0) - 2\phi'(0^+) \\ &= cm + 2\sqrt{tm^2 + \frac{u}{2}m^4} \qquad (t > 0) \end{aligned} \tag{9}$$

where we used m to denote $\phi(0)$. Setting $h = 0$, we see that Eq. 9 has a single real solution $m = 0$ for $c > 0$. For $c < 0$, taking $m \neq 0$ we have

$$c^2 = 4t + 2um^2 \tag{10}$$

which produces a non-zero real solution for m when $c^2 > 4t$, or $-c \geq 2\sqrt{t}$. This is the defect analogue of the surface transition expected for large negative c. For $t < 0$, $\phi(\pm\infty) = \sqrt{-t/u}$ so we choose $k = t^2/4u$ in Eq. 8 so that $\phi'(x_\perp \to \pm\infty) = 0$ again. Using this in Eq. 8, we have

$$(\phi')^2 = t\phi^2 + \frac{u}{2}\phi^4 + \frac{t^2}{2u} = \sqrt{u/2}(\phi^2 + t/u)^2 \tag{11}$$

For $x_\perp > 0$, ϕ' is negative if $\phi^2 > -t/u$ and vice versa so that we choose the minus sign when taking the square root. For $x_\perp < 0$, by the same argument we must pick the plus sign for the square root. So we have

$$\phi'(x_\perp) = -\mathrm{sgn}(x_\perp)\sqrt{u/2}(\phi(x_\perp)^2 + t/u) \tag{12}$$

Plugging this into Eq. 7, we find

$$h = cm + \sqrt{2u}(m^2 + t/u) \qquad (t < 0) \tag{13}$$

Setting $h = 0$, we have the quadratic equation for m,

$$m^2 + \frac{c}{\sqrt{2u}}m - \frac{|t|}{u} = 0 \tag{14}$$

from which one gets non-zero value for m. So the defect is ordered for all values of c when $t < 0$. Thus we see that, at the saddle point level at least, the analysis of $D = 1$ defects closely follows that of surfaces as expected.

For $D > 1$, assuming $SO(D)$ rotational symmetry around the defect simplifies Eq. 4 to

$$\phi'' + \frac{D-1}{r_\perp}\phi' = t\phi + u\phi^3 \quad (r_\perp \neq 0) \tag{15}$$

where the primes denote derivatives w.r.t. $r_\perp$. This is a nonlinear Poisson equation, for which we are not aware of any closed form solutions. Perhaps one may make progress with numerics, but the unusual boundary condition makes an immediate resolution difficult. We can make some preliminary arguments about the phase diagram, based on the boundary condition Eq. 5 at $h = 0$,

$$\lim_{\epsilon \to 0} \epsilon^{D-1}\phi'(\epsilon) = \frac{c}{S_D}\phi(0) \tag{16}$$

From this equation we can see that if the field is to have an enhanced value at the defect compared to the bulk, ϕ' should be negative near the defect. Due to the the magnetic field h, taken to 0 from above, $\phi(0)$ is of course non-negative. As a result, in this scenario, the signs of the two sides of Eq. 16 can match only if $c < 0$. We note that this is only a necessary condition. Similarly, we can see that if the profile of ϕ rises away from the defect, the matching of signs will necessitate $c > 0$. This argument shows only that a surface transition for negative c should persist even for $D > 1$. Unfortunately, we don't have more detailed results for the phase diagram at present.

II. LARGE N APPROXIMATION

We now consider the large N limit of the problem. To allow the limit to be taken consistently, we will define $u = \frac{u_0}{N}$ in Eq. 2. Following Ref. [7], let us first work out the propagator for the quadratic part of the Landau-Ginzburg Hamiltonian,

$$\beta\mathcal{H}_0 = \int \mathrm{d}^d x \vec{\phi}(x) \cdot \left[t - \nabla^2 + c\delta^{(D)}(x_\perp)\right]\vec{\phi}(x) \tag{17}$$

An even simpler task is to evaluate the propagator of the $c = 0$ case, which corresponds to the Gaussian theory without a defect. In Fourier space, this is given by

$$\beta\mathcal{H}_0^{(0)} = \int đ^d k_\perp \int đ^{d-D} k_\parallel \frac{t + k_\parallel^2 + k_\perp^2}{2}\vec{\phi}(k_\parallel, k_\perp)^2 \tag{18}$$

The Green's function for this theory can be read off easily,

$$\begin{aligned} g_{k_\parallel}^{(0)}(x_\perp, x'_\perp) &= \int đ^D k_\perp \frac{e^{ik_\perp\cdot(\vec{x}_\perp - x'_\perp)}}{t + k_\parallel^2 + k_\perp^2} \\ &= \frac{1}{(2\pi)^{D/2}}\left(\frac{\kappa}{|x_\perp - x'_\perp|}\right)^{\frac{D-2}{2}} K_{\frac{D-2}{2}}\left(\kappa\,|x_\perp - x'_\perp|\right) \end{aligned} \tag{19}$$

where $\kappa^2 = t + k_\parallel^2$, and we are implicitly assuming $t > 0$ for the Gaussian theory to be well-defined as usual. In the above integral, the limits of integration were left unbounded. This is fine for $D < 2$, where the integral does not need regularization; we can allow $k_\perp$ to run to infinitely large values without $g_{k_\parallel}^{(0)}(x_\perp, x_\perp)$ blowing up anywhere, e.g.

$$g_{k_\parallel}^{(0)}(x_\perp, x'_\perp) = \frac{1}{2\kappa}e^{-\kappa|x_\perp - x'_\perp|} \quad (D = 1) \tag{20}$$

But for $D \geq 2$, we need to incorporate a UV cutoff,

$$g_{k_\parallel}^{(0)}(x_\perp, x'_\perp) = \int^\Lambda đ^D k_\perp \frac{e^{ik_\perp\cdot(\vec{x}_\perp - x'_\perp)}}{\kappa^2 + k_\perp^2} \tag{21}$$

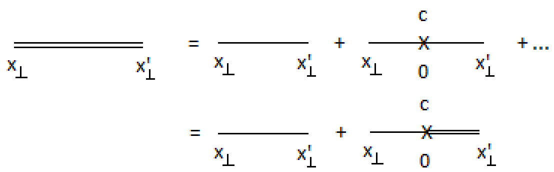

FIG. 1. Diagrammatic equation for $g_{k_\parallel}(x_\perp, x'_\perp)$

This matches the function in Eq. 19 for large $|x_\perp - x'_\perp|$ but must be amended for $|x_\perp - x'_\perp| \to 0$, by including the effect of the UV cutoff. Now the coupling c can be included using a series expansion.[7] This is best expressed diagrammatically, as shown in Fig. 1. In this figure, the double lines stand for $g_{k_\parallel}(x_\perp, x'_\perp)$, and the single lines stand for $g^{(0)}_{k_\parallel}(x_\perp, x'_\perp)$. This results in the following expression for the Green's function of $\beta\mathcal{H}_0$,

$$g_{k_\parallel}(x_\perp, x'_\perp) = g^{(0)}_{k_\parallel}(x_\perp, x'_\perp) - \frac{c g^{(0)}_{k_\parallel}(x_\perp, 0) g^{(0)}_{k_\parallel}(0, x'_\perp)}{1 + c g^{(0)}_{k_\parallel}(0,0)} \tag{22}$$

where $g^{(0)}_{k_\parallel}(0,0)$ is given by

$$g^{(0)}_{k_\parallel}(0,0) = \int^{\Lambda} đ^D k_\perp \frac{1}{\kappa^2 + k_\perp^2} \tag{23}$$

$$\approx \begin{cases} \frac{\Gamma(1-D/2)}{(4\pi)^{d/2}} \kappa^{D-2} & (D < 2) \\ \frac{1}{4\pi} \ln\left(1 + \frac{\Lambda^2}{\kappa^2}\right) & (D = 2) \\ \frac{S_D \Lambda^{D-2}}{(D-2)(2\pi)^D} & (D > 2) \end{cases} \tag{24}$$

We have taken the UV cutoff to ∞ for $D < 2$. The next step is to incorporate the ϕ^4 term to evaluate the full Green's function $G_{k_\parallel}(x_\perp, x'_\perp)$, defined as

$$G_{k_\parallel}(x_\perp, x'_\perp) = \int \mathrm{d}^{d-D} x_\parallel e^{i k_\parallel \cdot x_\parallel} \langle \phi_i(x_\parallel, x_\perp) \phi_i(0, x'_\perp) \rangle \tag{25}$$

Of course, doing this in general is hard, so we look for approximations. In the limit of large N, the ϕ^4 term can be incorporated at all orders in perturbation theory in a self-consistent manner. The leading order result for $N = \infty$ includes only the graphs with maximal number of loops. This is expressed using diagrams in Fig. 2.[7] The corresponding equation for the Green's function is

$$G_{k_\parallel}(x_\perp, x'_\perp) = g_{k_\parallel}(x_\perp, x'_\perp) - \int d^D y_\perp V(y_\perp) \times g_{k_\parallel}(x_\perp, y_\perp) G_{k_\parallel}(y_\perp, x'_\perp) \tag{26}$$

where $V(x_\perp)$ is given by the (appropriately regulated) local component of the Green's function

$$V(x_\perp) = u_0 \int_{|p_\parallel| < \Lambda} đp_\parallel \left(G_{p_\parallel}(x_\perp, x_\perp) - G_{p_\parallel}(\infty, \infty)\right) \tag{27}$$

The pair of functions $G_{k_\parallel}(x_\perp, x'_\perp)$ and $V(y_\perp)$ must be calculated self-consistently. We have not succeeded in completing this line of analysis, so we postpone it to future work.

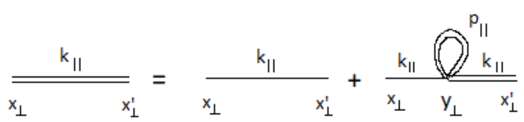

FIG. 2. Diagrammatic equation for $G_{k_\parallel}(x_\perp, x'_\perp)$ (double lines). Single lines stand for $g_{k_\parallel}(x_\perp, x'_\perp)$.

III. PLANE DEFECTS IN $d = 4$

Plane defects in 4d correspond to $D = 2$. The corresponding saddle point equation is

$$\phi'' + \frac{1}{r}\phi' = t\phi + u\phi^3 \tag{28}$$

where we have replaced $r_\perp$ with r for less cumbersome notation. Substituting $\phi(r) = \frac{f(r)}{\sqrt{ur}}$ in Eq. 28, we have

$$f'' - \frac{1}{r} f' + \frac{1}{r^2} f = t f + \frac{f^3}{r^2} \tag{29}$$

The boundary condition Eq. 5 for ϕ translates to the following boundary condition for f:

$$\lim_{\epsilon \to 0} \left[(1 + c)\frac{f(\epsilon)}{\epsilon} - f'(\epsilon)\right] = h \tag{30}$$

Eq. 29 involves a length scale $|t|^{-1/2}$. If we send this scale to infinity by setting $t = 0$, the equation achieves a scaling symmetry $f(r) \to f(\lambda r)$:[1]

$$f'' - \frac{1}{r} f' + \frac{1}{r^2}(f - f^3) = 0 \tag{31}$$

A change of variables $s = -\ln r$ turns the above equation into a Duffing equation without a forcing term:[8]

$$\frac{d^2 f}{ds^2} + 2\frac{df}{ds} + f - f^3 = 0 \tag{32}$$

The $s \to \infty$ limit corresponds to the defect $r = 0$, and the $s \to -\infty$ limit corresponds to the bulk. Since we set $t = 0$, we must have $\phi = 0$ in the bulk, which implies the boundary condition

[1] We thank Max Metlitski for pointing this out.

$\lim_{s\to-\infty} e^s f(s) = 0$. In the language of the Duffing oscillator, s is to be interpreted as a time coordinate. For the above equation, the "oscillator" has unstable equilibria at $f = \pm 1$ and a stable equilibrium at $f = 0$.[9] For generic initial condition $f(-\infty)$, the oscillator is known to approach the stable equilibirum $f(\infty) = 0$. This is not enough since we need to know the limit of $f(s)e^s$ as $s \to \infty$ in order to figure out the behavior of the order parameter field $\phi(r)$ at $r = 0$. In order to do so, one needs to do asymptotic analysis of Eq. 32.

Lastly, let us make some brief comments about the extraordinary transition for plane defects in $d = 4$. Following Ref. [10], the theory to be considered may be described by the action $S_{\text{ordinary}} + S_n + S_{n\phi}$, where S_{ordinary} is the usual $O(N)$ model in terms of the bulk $\vec{\phi}(x)$ field, S_n is the nonlinear sigma model in terms of the N component field $\vec{n}(x_\parallel)$ defined on the defect along with a symmetry-breaking field included as a regulation scheme,

$$S_n = \int d^2x_\parallel \left(\frac{1}{2g} \left(\nabla_\parallel \vec{n}\right)^2 - \vec{h}\cdot\vec{n} \right) \tag{33}$$

and $S_{n\phi}$ is the coupling of the fields $\vec{\phi}(x)$ and $\vec{n}(x_\parallel)$. For $g \to 0$, the fluctuations of $\vec{n}$ are frozen so that it behaves like a symmetry-breaking field for the bulk. This should take the theory to a defect analogue of the normal fixed point known in the case of surfaces. An alternative description was proposed in Ref. [10], in which one considers the normal fixed point as the reference theory and restores $O(N)$ symmetry order by order in g. For plane defects, we have

$$S = S_{\text{normal}} + S_n - s \int d^2x_\parallel n_i(x_\parallel)\hat{\phi}_i(x_\parallel) + \delta S \tag{34}$$

where $i = 1, \cdots, N-1$ and δS includes terms that are required to recover an $O(N)$ symmetric action. Here we have made the assumption that analogous to the bulk-surface OPE (operator product expansion) considered in Ref. [10], we have a bulk-defect OPE that relates $\phi_i(x)$ $(i = 1, \cdots, N-1)$ to the $O(N-1)$ vector $\hat{\phi}_i(x_\parallel)$ as

$$\phi_i(x_\parallel, x_\perp) \sim \mu_\phi |x_\perp|^{d-2-\Delta_\phi} \hat{\phi}_i(x_\parallel) + \cdots, \quad |x_\perp| \to 0 \tag{35}$$

In other words, we conjecture that the analogous $\hat{\phi}_i$ in this case has dimension $d-2$. For the bulk-defect OPE of the operator $\phi_N \equiv \sigma$, our conjecture is that the suitable generalization of $\hat{\sigma}$ is the so-called displacement operator, with scaling dimension $d - D + 1 = 4 - 2 + 1 = 3 > 2$.[11] This means that the coupling of $\hat{\sigma}$ to n_N is irrelevant at the $g = 0$ fixed point. We have not been able to formulate a proof for the above claims about the OPEs. But if they are true, we find that the arguments in Ref. [10] also apply to plane defects in $d = 4$, resulting in an "extraordinary log" phase of the defect for small enough N.

IV. CONCLUSION

In this paper, we have attempted to capture the physics of defects in the $O(N)$ model. Admittedly, our analysis for $D > 1$ defects is largely incomplete. However, our primary aim here has been to set up new ways of thinking about this problem. The missing details in our analysis are indicative of promising directions for exploration in future work.

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[1] K Binder and PC Hohenberg. Surface effects on magnetic phase transitions. *Physical Review B*, 9(5):2194, 1974.

[2] T C Lubensky and Morton H Rubin. Critical phenomena in semi-infinite systems. II. Mean-field theory. Technical report, 1975.

[3] Kaoru Ohno and Yutaka Okabe. The 1/n expansion for the extraordinary transition of semi-infinite system. *Progress of theoretical physics*, 72(4):736–745, 1984.

[4] Theodore W Burkhardtt and John L Cardys. Surface critical behaviour and local operators with boundary-induced critical profiles. Technical report, 1987.

[5] RZ Bariev and IZ Ilaldinov. Correlation functions of the n-vector model with a large-scale defect in the limit n to infinity. *Journal of Physics A: Mathematical and General*, 22(18):L879, 1989.

[6] R. Abe. d'-Dimensional Defect in d-Dimensional Lattice. II: Local Exponents β', γ' and ν' in the Limit $n \to \infty$. *Progress of Theoretical Physics*, 65(6):1835–1843, jun 1981.

[7] A. J. Bray and M. A. Moore. Critical behaviour of semi-infinite systems. *Journal of Physics A: General Physics*, 10(11):1927–1962, nov 1977.

[8] Eric W Weisstein. Duffing differential equation. *https://mathworld. wolfram. com/*, 2003.

[9] Takashi Kanamaru. Duffing oscillator. *Scholarpedia*, 3(3):6327, 2008.

[10] Max A Metlitski. Boundary criticality of the O(N) model in d = 3 critically revisited. Technical report, 2020.

[11] Marco Billo, Vasco Gonçalves, Edoardo Lauria, and Marco Meineri. Defects in conformal field theory. *Journal of High Energy Physics*, 2016(4):91, 2016.