Thermodynamics of Ideal Type IIB Superstring Gases and Early-Universe Cosmology

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The current inflationary theory of the early universe is singular when the effects of quantum gravity are expected to become large. Since string theory is a promising candidate for a theory of quantum gravity, it makes sense to consider a model of the early universe based on strings. *String gas cosmology* is exactly such a model: The early universe is considered to be a gas of ideal strings at high temperature. In this paper, we concentrate specifically on the thermodynamics of a gas of closed Type IIB superstrings. Closely following the approach in [6], we compute the one-loop partition function and derive the so-called *Hagedorn temperature* of this theory (above which the canonical partition function becomes ill-defined). We also analyze the phase structure of this theory in a universe with compactified spatial dimensions, and we show following [3] that the apparent temperature singularity in the classical inflationary theory is resolved.

I. INTRODUCTION

The cosmology of the early universe remains an important question in high-energy physics. In the current paradigm, the accepted theory of early-universe cosmology is that of *inflation*. It is expected that some scalar field with negative energy density dominated at early times, driving an accelerated expansion of space [5]. The inflationary theory explains several notable features of our universe, including its apparent homogeneity, isotropy, and flatness (zero curvature).

Nevertheless, the inflationary paradigm presents some problematic features. Most importantly, inflationary theory in the context of classical general relativity necessarily predicts [2] the existence of an initial cosmological singularity: An infinitely small, hot, and dense initial state of the universe. Such a singularity is an an artifact; in particular, the effective field theory approach used to describe inflation is expected to break down anyway at the Planck scale [3]. Thus, we expect that within a proper theory of quantum gravity, the initial cosmological singularity should be resolved.

In recent years, superstring theory has presented itself as a candidate theory of quantum gravity [4]. Thus, an effort has been made to investigate early-universe cosmology in the context of string theory. It was hoped that string theory, as a candidate theory of quantum gravity, could resolve many of the problems plaguing the picture provided by inflation, most importantly the singularity.

To make progress towards a theory of early-universe cosmology based on strings, it becomes essential to consider the behavior of a high-temperature ideal string gas [6]. String gases have a number of interesting and unique properties. Perhaps most importantly, the density of oscillatory string modes increases exponentially with energy, so there actually exists a critical temperature T_H, the so-called Hagedorn temperature, beyond which the string gas partition function is no longer well-defined [6]. The precise critical behavior of the string gas near T_H remains poorly understood and continues to be an active area of research.

In this paper, we will discuss the thermodynamics of an ideal gas of closed Type IIB strings. We will use the methods of [6] to calculate of the torus partition function at finite temperature for the Type IIB superstring (by contrast, the subject of [6] was primarily the $SO(32)$ heterotic string). We calculate the Hagedorn temperature T_H^{IIB} of the Type IIB string, and we determine the phase structure of the Type IIB string at temperatures $T < T_H^{\text{IIB}}$ in a toroidally compactified spacetime. Finally, we use our previous results to show, following [3], that there is no temperature singularity in the string gas model.

II. THE STRING GAS

Consider a gas of ideal closed strings at finite temperature. All relevant thermodynamic quantities can be computed from the finite-temperature partition function

$$\mathcal{Z} = \text{tr}(e^{-\beta\mathcal{H}}). \tag{1}$$

Now, the partition function can be expressed as a sum over vacuum string amplitudes:

$$\mathcal{Z} = \exp\left(Z_0 + Z_1 + \cdots\right), \tag{2}$$

where $\mathcal{M}_g$ is the vacuum string amplitude of genus g. The terms $\cdots$ at higher genus are then at higher order in the string coupling. The sphere amplitude Z_0 is a trivial constant factor [7], so the first nontrivial contribution to the string partition function at lowest order in the string coupling is the thermal torus amplitude Z_1. We can thus write

$$\log\mathcal{Z} = Z(\beta) + \cdots . \tag{3}$$

where we have relabeled $Z(\beta) \equiv Z_1$. The study of the string gas is therefore reduced to the computation of

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the thermal torus vacuum amplitude (partition function) $\mathcal{Z}(\beta)$ at finite temperature.

III. ONE-LOOP STRING PARTITION FUNCTION

We now wish to calculate the one-loop partition function at finite temperature for a Type IIB superstring. This calculation is lengthy, and for brevity and ease of understanding, we will not show all of the details. We will simply elucidate the steps we take by working through the much simpler case of the bosonic string. We will then quote the final result for the Type IIB superstring and note the relevant discrepancies. Throughout our calculation, we use units where the string slope parameter $\alpha' = 1$.

A. Zero Temperature

Before calculating the finite-temperature torus partition function, we consider the case of zero temperature. Consider a closed bosonic string in D dimensions at zero temperature. We calculate the torus partition function as follows. Consider a torus with modulus τ. We can construct a CFT on this torus by taking a free CFT on a circle, translating by $2\pi\tau = 2\pi(\tau_1 + i\tau_2)$, and identifying the ends [7]. This amounts to a translation in Euclidean time by $2\pi\tau_2$ and translating in the spatial parameter σ_1 by $2\pi\tau_1$. Tracing, we find

$$Z_{\rm b}(\tau) = {\rm tr}\left[e^{2\pi i\tau_1 P - 2\pi\tau_2 H}\right] = (q\overline{q})^{-d/24}{\rm tr}(q^{L_0}\overline{q}^{\tilde{L_0}}), \quad (4)$$

where as usual $q = e^{2\pi i\tau}$. Note that conservation of momentum requires the following level-matching:

$$p_L = p_R \equiv p \quad (5)$$

Here p_L, p_R are the left- (right)-moving momenta of the string. After tracing over the entire string spectrum (integrating over the string momentum and summing over the occupation numbers of the oscillatory modes) and using the level-matching condition, we obtain

$$Z_{\rm b}(\tau) = V_d(q\overline{q})^{-d/24}\sum_{p_R,p_L} q^{p_L^2/4}\overline{q}^{p_R^2/4}\prod_{i,n}\sum_{N_{i,n},\tilde{N}_{i,n}} q^{nN_{i,n}}\overline{q}^{n\tilde{N}_{i,n}} = V_d(q\overline{q})^{-d/24}\int\frac{d^dp}{(2\pi)^d}e^{-\pi\tau_2 p^2}\prod_{i,n}\sum_{N_{i,n},\tilde{N}_{i,n}} q^{nN_{i,n}}\overline{q}^{n\tilde{N}_{i,n}}. \quad (6)$$

Here $N_{i,n}, \tilde{N_{i,n}}$ are the occupation numbers of the nth left- (right)-moving oscillatory mode of the string spectrum, and i is an index running over the $d-2$ transverse modes of the string. Performing the summations and integrations, we find the result

$$Z_{\rm b}(\tau) = V_d(2\pi)^{-d}\tau_2^{-d/2}\left|\eta(\tau)\right|^{-2D}, \quad (7)$$

where $D = d-2$ and V_d is the volume of d-dimensional spacetime. The total torus partition function may be found by integrating the invariant measure $d\tau_1 d\tau_2/(2\tau_2)$ over the fundamental domain $\mathscr{F}$ of the torus moduli space. In $d = 26$ dimensions, we have

$$Z_{\rm b} = \frac{V_{26}}{2(2\pi)^{26}}\int_{\mathscr{F}}\frac{d\tau_1 d\tau_2}{\tau_2^{14}}\left|\eta(\tau)\right|^{-48}. \quad (8)$$

By a more involved but similar calculation, we can compute the zero-temperature torus partition function for the Type IIB superstring in $d = 10$ dimensions:

$$Z_{\rm IIB} = \frac{V_{10}}{2(2\pi)^{10}}\int_{\mathscr{F}}\frac{d\tau_1 d\tau_2}{\tau_2^{6}}\left|\eta(\tau)\right|^{-16}\sum_{a,b,\overline{a},\overline{b}\in\{0,1\}}(-1)^{a+b+ab}(-1)^{\overline{a}+\overline{b}+\overline{a}\overline{b}}\frac{\vartheta\left[{a \atop b}\right]^4}{\eta(\tau)^4}\frac{\overline{\vartheta}\left[{\overline{a} \atop \overline{b}}\right]^4}{\overline{\eta}(\tau)^4}. \quad (9)$$

Here $\vartheta\left[{a \atop b}\right]$ is the Jacobi ϑ-function.

B. Finite Temperature

We now move to the case of finite temperature. Again, we will illustrate the crucial features of the IIB superstring computation by considering the bosonic string computation only. It is familiar from quantum field theory that the thermal partition function at inverse temperature β of a QFT in d dimensions is equal to the Euclidean partition function in d dimensions with the time dimension compactified on a S^1 of period $\beta = 2\pi R$. As it turns out, the same analogy holds true in the case of the bosonic string. It thus suffices to compute the torus partition function for a free d-dimensional bosonic string with compactified Euclidean time.

To proceed, consider a single compactified dimension, and let p_L, p_R be the momenta along this direction. The level-matching condition is then modified by the compactified dimension. In particular, the left- and right-moving momenta can take the following form [7]:

$$p_L = \frac{m}{R} - nR, \qquad p_R = \frac{m}{R} + nR. \tag{10}$$

Here m is the wavenumber of the momentum along the compactified direction and n is the winding number of the string. Using our zero-temperature result from the previous section, the full partition function with $d-1$ uncompactified dimensions and compact Euclidean time with period $\beta = 2\pi R$ is then

$$Z_{\mathrm{b}}(\beta;\tau) = V_{d-1} \int \frac{d^{d-1}p}{(2\pi)^{d-1}} e^{-\pi\tau_2 p^2} \sum_{p_L,p_R} q^{p_L^2/4} \overline{q}^{p_R^2/4} \prod_{i,k} \sum_{N_k,\tilde{N}_k} q^{nN_{i,k}} \overline{q}^{n\tilde{N}_{i,k}}. \tag{11}$$

Recalling $q = e^{2\pi i \tau}$ and sing the Poisson summation formula, we can compute

$$\sum_{p_L,p_R} q^{p_L^2/4} \overline{q}^{p_R^2/4} = \frac{R}{\sqrt{\tau_2}} \sum_{\tilde{m},n\in\mathbb{Z}} \exp\left(-\frac{\pi R^2}{\tau_2} \left|\tilde{m} - n\tau\right|^2\right). \tag{12}$$

Putting all of the pieces together, we find

$$Z_{\mathrm{b}}(\beta;\tau) = \beta V_{d-1} (2\pi)^{-d} \tau_2^{-d/2} \left|\eta(\tau)\right|^{-2D} \sum_{\tilde{m},n\in\mathbb{Z}} \exp\left(-\frac{\pi R^2}{\tau_2} \left|\tilde{m} - n\tau\right|^2\right). \tag{13}$$

Integrating over the moduli space for τ in $d = 26$ dimensions, we finally arrive at the finite-temperature torus partition function for the bosonic string:

$$Z_{\mathrm{b}}(\beta) = \frac{\beta V_{25}}{2(2\pi)^{26}} \sum_{\tilde{m},n\in\mathbb{Z}} \int_{\mathscr{F}} \frac{d\tau_1 d\tau_2}{\tau_2^{14}} \frac{e^{-\frac{\pi R^2}{\tau_2}\left|\tilde{m}-n\tau\right|^2}}{\left|\eta(\tau)\right|^{48}}. \tag{14}$$

The case of the Type IIB superstring is slightly more complicated, as the introduction of finite temperature is no longer a simple toroidal compactification of Euclidean time (as a matter of fact, it turns out to be a slightly more complicated orbifold compactification). Nevertheless, we may write down the expression for the thermal partition function at finite temperature β for the Type IIB string in $d = 10$ dimensions as below [6]:

$$Z_{\mathrm{IIB}}(\beta) = \frac{\beta V_9}{2(2\pi)^{10}} \sum_{\tilde{m},n\in\mathbb{Z}} \int_{\mathscr{F}} \frac{d\tau_1 d\tau_2}{\tau_2^6} \sum_{a,b,\overline{a},\overline{b}\in\{0,1\}} (-1)^{(a+\overline{a})\tilde{m}+(b+\overline{b})n} \frac{e^{-\frac{\pi R^2}{\tau_2}\left|\tilde{m}-n\tau\right|^2}}{\left|\eta(\tau)\right|^{16}} (-1)^{a+b+ab} (-1)^{\overline{a}+\overline{b}+\overline{a}\overline{b}} \frac{\vartheta\left[\begin{smallmatrix} a \\ b \end{smallmatrix}\right]^4}{\eta(\tau)^4} \frac{\overline{\vartheta}\left[\begin{smallmatrix} \overline{a} \\ \overline{b} \end{smallmatrix}\right]^4}{\overline{\eta}(\tau)^4}. \tag{15}$$

The fundamental difference between Eqn. (15) and the bosonic partition function in Eqn. (14) is the factor $(-1)^{(a+\overline{a})\tilde{m}+(b+\overline{b})n}$. Roughly speaking, this arises due to the antiperiodicity of the fermionic degrees of freedom of the Type IIB superstring in the compactified Euclidean time corresponding to finite temperature–in other words, we see that finite temperature is a twisted orbifold compactification as opposed to a regular toroidal compactification.

IV. THE HAGEDORN TEMPERATURE OF TYPE IIB SUPERSTRINGS

Having calculated the torus partition function for the Type IIB superstring, we can in fact derive the Hagedorn temperature. Although it is not immediately apparent from Eqn. (15), the torus partition function for the Type IIB superstring is not well-defined for all temperatures β. In fact, $Z_{\mathrm{IIB}}(\beta)$ suffers a singularity as β approaches a fixed value β_H^{IIB} from above; this value is the (inverse) Hagedorn temperature. Our goal in this section is to use our expression for $Z_{\mathrm{IIB}}(\beta)$ to derive a formula for the Hagedorn temperature.

To proceed, we follow [6] and make the following definitions:

$$\begin{aligned} V_8 &= \frac{\vartheta\left[\begin{smallmatrix}0\\0\end{smallmatrix}\right]^4 - \vartheta\left[\begin{smallmatrix}0\\1\end{smallmatrix}\right]^4}{2\eta^4}; \\ S_8 &= \frac{\vartheta\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]^4 + \vartheta\left[\begin{smallmatrix}1\\1\end{smallmatrix}\right]^4}{2\eta^4} \end{aligned}. \tag{16}$$

We then verify by direct calculation that

$$V_8 - (-1)^{\tilde{m}} S_8 = \frac{1}{2}\sum_{a,b} (-1)^{\tilde{m}a}(-1)^{a+b+ab}\frac{\vartheta\left[\begin{smallmatrix}a\\b\end{smallmatrix}\right]^4}{\eta(\tau)^4}. \tag{17}$$

We now return to the formula for $Z_{\mathrm{IIB}}(\beta)$. We notice that there is a sum over the integers $\tilde{m}$ and n. Now, the integrand is modular invariant under a $SL_2(\mathbb{C})$ transformation of τ (as we expect). Thus, we can identify several of the summands corresponding to pairs $(\tilde{m}, n)$ using a $SL_2(\mathbb{C})$ transformation. In particular, we can reduce the sum over $(\tilde{m}, n)$ into the $SL_2(\mathbb{C})$ orbit of $(0, 0)$, integrated over the fundamental domain $\mathscr{F}$, along with the orbits of $(\tilde{m}, 0)$ for $\tilde{m} \neq 0$, integrated over the region $\sqcup = \left\{(\tau_1, \tau_2) \mid -\frac{1}{2} \le \tau_1 \le \frac{1}{2},\ \ \tau_2 \ge 0\right\}$. Using the definitions of V_8, S_8 to simplify our expression, we have

$$Z_{\mathrm{IIB}}(\beta) = \frac{2\beta V_9}{(2\pi)^{10}}\int_{\mathscr{F}}\frac{d\tau_1 d\tau_2}{\tau_2^6}\frac{\left|V_8 - S_8\right|^2}{\left|\eta(\tau)\right|^{16}} + \frac{2\beta V_9}{(2\pi)^{10}}\int_{\sqcup}\frac{d\tau_1 d\tau_2}{\tau_2^6}\sum_{\tilde{m}\neq 0} e^{-\frac{\pi R^2}{\tau_2}\tilde{m}^2}\frac{\left|V_8 - (-1)^{\tilde{m}} S_8\right|^2}{\left|\eta(\tau)\right|^{16}}. \tag{18}$$

By the properties of the Jacobi ϑ-function, we may show [7] that $V_8 - S_8 = 0$. Thus, the first term in Eqn. (18) vanishes, and the second term reduces to a sum over odd integers $\tilde{m} = 2\tilde{\jmath} + 1$:

$$Z_{\mathrm{IIB}}(\beta) = \frac{2\beta V_9}{(2\pi)^{10}}\int_0^\infty \frac{d\tau_2}{\tau_2^6}\int_{-1/2}^{1/2} d\tau_1 \sum_{\tilde{\jmath}\in\mathbb{Z}} e^{-\frac{\pi R^2}{\tau_2}(2\tilde{\jmath}+1)^2}\frac{\left|V_8 + S_8\right|^2}{\left|\eta(\tau)\right|^{16}}. \tag{19}$$

At this point, it is convenient to use the q-expansions of $V_8 + S_8$. We can numerically compute [6] that

$$\frac{V_8 + S_8}{\eta^8} = \sum_{K=0}^\infty S_K q^K, \quad \frac{\overline{V}_8 + \overline{S}_8}{\overline{\eta}^8} = \sum_{\overline{K}=0}^\infty \overline{S}_{\overline{K}}\overline{q}^{\overline{K}} \tag{20}$$

where $S_K = 16, 256, 2304, \cdots$ for $K = 0, 1, 2, \cdots$ respectively. Note that this q-expansion contains no negative powers of q–this reflects the familiar fact that there are no tachyons in the Type IIB superstring spectrum. Substituting these series in we see that the integrand depends on τ_1 as $e^{2\pi i\tau_1(K-\overline{K})}$. We thus see that all terms with $\overline{K} \neq K$ are annihilated, and integration over τ_1 yields 1 for all terms with $\overline{K} = K$ We thus find

$$Z_{\mathrm{IIB}}(\beta) = \frac{2\beta V_9}{(2\pi)^{10}}\sum_{K=0}^\infty\sum_{\tilde{\jmath}\in\mathbb{Z}}\left|S_K\right|^2\int_0^\infty\frac{d\tau_2}{\tau_2^6}e^{-\frac{\pi R^2}{\tau_2}(2\tilde{\jmath}+1)^2 - 4\pi\tau_2 K} \equiv \beta V_9\beta^{-10}\sum_{K\ge 0}\left|S_K\right|^2 G_{10}(2\beta\sqrt{K}). \tag{21}$$

Here the function $G_{10}(2\beta\sqrt{K})$ is found by doing the integration over τ_2; the result involves a sum over Bessel functions. At low temperatures ($\beta \gg 1$), we also have the asymptotic limit

$$G_d(x) \simeq 2\left(\frac{x}{2\pi}\right)^{\frac{d-1}{2}} e^{-x}, \quad x \gg 1. \tag{22}$$

From this, we can now determine an expression for the Hagedorn temperature. Consider the coefficients S_K, determined from the q-expansion in Eqn. (20). As determined in [1], these coefficients increase exponentially as

$$S_K \sim (2\sqrt{K})^{-11/2}\exp\left(2\pi\sqrt{2K}\right). \tag{23}$$

Using the asymptotic behavior of $G_d(x)$, we thus see that the summands in the right hand side of Eqn. (21) go as $\sim \exp(4\pi\sqrt{2K} - 2\beta\sqrt{K})$. The sum therefore converges only for $\beta > \beta_H^{\mathrm{IIB}}$, where

$$\beta_H^{\mathrm{IIB}} = 2\pi\sqrt{2}. \tag{24}$$

This is precisely the Hagedorn temperature of the Type IIB superstring; for $\beta < \beta_H^{\mathrm{IIB}}$, our expression for the canonical partition function $Z_{\mathrm{IIB}}(\beta)$ ceases to make sense anymore.

This is a remarkable result. It sets a domain of validity for the canonical ensemble, beyond which it becomes very difficult to discuss the string behavior. Indeed, the exact behavior of the Type IIB superstring gas beyond the Hagedorn temperature is still an open question [6].

V. TOROIDAL COMPACTIFICATION AND PHASE TRANSITIONS

When studying string gases in the context of the early universe, it becomes necessary to consider systems with a very small spatial extent. We can then ask the following question: How does our ideal Type IIB superstring gas behave in a spacetime with spatial dimensions compactified. To simplify our analysis, let us consider a Type IIB superstring gas in $d = 10$ at finite (inverse) temperature $\beta = 2\pi R$, and suppose that the spatial dimension X_9 is also compactified with period $2\pi R_9$. What does the one-loop torus partition function $Z(\beta)$ look like now?

To answer this question, we recall the effect of toroidal compactification on the momenta p_L^9, p_R^9 in the X_9 direction. These momenta are discretized, classified by their wavenumber m and winding number n as per Eqn. (10). We then have

$$(4\pi^2\tau_2)^{-1/2} = \int \frac{dp}{(2\pi)} e^{-\pi\tau_2 p_9^2} \to \sum_{m,n} q^{(p_L^9)^2/4}\overline{q}^{(p_R^9)^2/4}, \tag{25}$$

where p_L, p_R are given by Eqn. (10) with compact dimension radius $R = R_9$. We also need to replace the spatial volume V_9 with the volume V_8 of the 8 remaining noncompact dimensions. Making these replacements in Eqn. (19), we then find

$$Z_{\rm IIB}(\beta) = \frac{2\beta V_8}{(2\pi)^9}\int_{\sqcup} \frac{d\tau_1 d\tau_2}{\tau_2^{11/2}} \sum_{\tilde{\jmath}\in\mathbb{Z}} e^{-\frac{\pi R^2}{\tau_2}(2\tilde{\jmath}+1)^2} \frac{|V_8+S_8|^2}{|\eta(\tau)|^{16}} \sum_{m,n} q^{\frac{1}{4}\left(\frac{m}{R_9}-nR_9\right)^2}\overline{q}^{\frac{1}{4}\left(\frac{m}{R_9}+nR_9\right)^2}. \tag{26}$$

Now, making use of the q-expansions in Eqn. (20) in Eqn. (26), we find that

$$Z_{\rm IIB}(\beta) = \frac{2\beta V_8}{(2\pi)^9} \sum_{K,\overline{K}\geq 0} \int_0^\infty \frac{d\tau_2}{\tau_2^{11/2}} \int_{-1/2}^{1/2} d\tau_1 \sum_{\tilde{\jmath}\in\mathbb{Z}} e^{-\frac{\pi R^2}{\tau_2}(2\tilde{\jmath}+1)^2} \sum_{m,n} q^{K+\frac{1}{4}\left(\frac{m}{R_9}-nR_9\right)^2}\overline{q}^{\overline{K}+\frac{1}{4}\left(\frac{m}{R_9}+nR_9\right)^2}. \tag{27}$$

Recalling $q = e^{2\pi i\tau}$, we see that the integrand depends on τ_1 as $e^{2\pi i\tau_1(K-\overline{K}-mn)}$. Hence, integration over τ_1 annihilates all terms with $K-\overline{K}\neq mn$ and yields 1 if $K-\overline{K} = mn$. Integrating the result over τ_2 yields an expression in terms of the same function $G_d(x)$ as in Eqn. (21). We find [6]

$$Z_{\rm IIB}(\beta) = \beta V_8 \beta^{-9} \sum_{m,n\in\mathbb{Z}} \sum_{K,\overline{K}\geq 0,\ K-\overline{K}=mn} S_K\overline{S}_{\overline{K}} G_9\left(\beta\sqrt{2K+2\overline{K}+\frac{m^2}{R_9^2}+n^2R_9^2}\right). \tag{28}$$

Admittedly, the result in Eqn. (28) is somewhat difficult to interpret. To make any further progress, we consider now the case of low temperature: $\beta > \beta_H^{\rm IIB}$. In this case, the series in Eqn. (28) is convergent. To determine the dominant behavior, we are searching for the largest contributing summands. These correspond to the lightest modes of our string. Note now that $K,\overline{K}\geq 0$. [Note: The condition $K,\overline{K}\geq 0$ is true *only* in the case of the Type IIB and IIA superstrings. In particular, as discussed in [6], it is *not* true for the heterotic string. In fact, there are additional light modes for the heterotic string [6].] We then find that the lightest modes come in two classes:

- Winding modes with $K = \overline{K} = m = 0$ and $n \neq 0$ are light when $R_9 \ll \beta^{-1}$ is small.

- Oscillatory modes with $K = \overline{K} = n = 0$ and $m \neq 0$ are light when $R_9 \gg \beta$ is large.

In addition, the mode $K = \overline{K} = m = n = 0$ is massless and will always contribute.

Now, we see from Eqn. (3) that the free energy density $\mathcal{F}_{\rm IIB}$ of the superstring gas is related to the quantity $Z_{\rm IIB}(\beta)$ via $\mathcal{F}_{\rm IIB} = -Z_{\rm IIB}/(\beta V_8)$. We can thus express $\mathcal{F}_{\rm IIB}$ as a piecewise function as follows:

$$\mathcal{F}_{\rm IIB}(T) = \begin{cases} -T^9\,|S_0|^2\left[G_9(0)+\sum_{m\neq 0}G_9(\beta m/R_9)\right] & R_9\gg T^{-1} \\ -T^9\,|S_0|^2\left[G_9(0)+\sum_{n\neq 0}G_9(\beta nR_9)\right] & R_9\ll T \\ -T^9\,|S_0|^2\,G_9(0) & \text{otherwise}\end{cases}. \tag{29}$$

The exact form of the function G_9 is irrelevant; we need only note that $G_d(x)$ is positive and decreasing. Thus, we see that $\mathcal{F}_{\rm IIB}$ attains a maximum for $T^{-1} \gtrsim R_9 \gtrsim T$, falling off identically in the limits $R \gg T^{-1}$ and $R \ll T$.

The structure above thus showcases a phase transition in the toroidal compactification parameter R_9. For the phase $R_9 \gg T^{-1}$ or $R_9 \ll T$, the free energy tends to a minimum for $R_9 \to \infty$ or $R_9 \to 0$, respectively, so the compactified dimension vanishes in these cases. In the phase $T^{-1} \gtrsim R_9 \gtrsim T$, the free energy is essentially constant, so the compactified dimension can attain any size R_9 within this range.

A few things above the above formula are worth noting. First of all, this formula differs from the formula for the heterotic string as derived in [6]. The key difference is that the heterotic string contains an additional phase for $R \simeq R^{-1}$ in which $\mathcal{F}$ attains a local minimum. Hence, the heterotic string tends to attract the compactified dimension size to the critical value $R = R^{-1}$, while the Type IIB superstring shows no preference for this point.

Second, this expression is symmetric under the interchange $R_9 \to R_9^{-1}$. This is actually a manifestation of a much larger symmetry of superstring theories in general: T-duality. Roughly speaking, T-duality relates a theory with a small compact dimension to a theory with a large compact dimension. (The precise formulation of T-duality actually relates the spectrum of a Type IIB superstring theory to a Type IIA theory [3], but the free energy will turn out the same in this specific limit.) This provides an important tool to relate small scales to large scales. As we will see later, this feature will prove extremely useful in resolving the initial singularity problem in early-universe cosmology.

VI. APPLICATIONS TO EARLY-UNIVERSE COSMOLOGY

Having sufficiently developed the theory of the Type IIB superstring gas, we now move to considering applications of this theory to early-universe cosmology. To begin, we consider an important question: How does the temperature of a string gas behave as the spatial extent of the gas is changed? This is an important question, and we are especially interested in the limit where the spatial extent of the string gas is brought down to zero. In the inflationary model coupled to classical gravity, the temperature is singular in the limit of zero spatial extent. We now investigate the corresponding behavior in the string gas model.

Since the Type IIB superstring gas has a Hagedorn temperature as computed in Section IV, we expect its temperature behavior to change as the Hagedorn temperature is approached. In particular, we cannot expect the temperature of the string gas to increase without bound beyond β_H as its spatial extent is decreased. We can use our expression in Eqn. (29) for the free energy density of the Type IIB superstring gas to gain an understanding of the gas temperature $T = \beta^{-1}$ as a function of the compact dimension size R_9. We analyze the three regimes

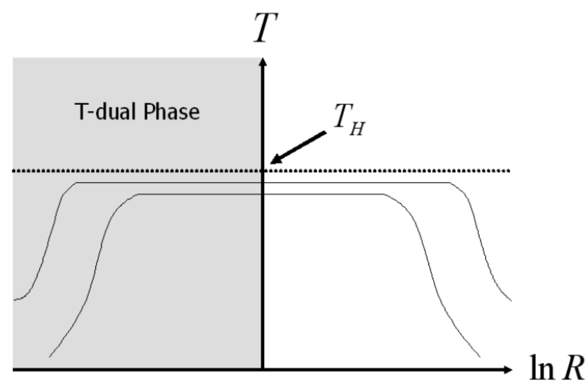

FIG. 1. A plot of temperature T as a function of compact dimension radius R (denoted R_9 in the main body of the paper). Note that the temperature approaches a fixed value $T < T_H^{\rm IIB}$ as the compact dimension size is decreased. This figure was originally produced in [3].

separately:

- $R_9 \gg T^{-1}$: From the free energy, we may compute the entropy of the universe $\mathcal{S}_{\rm IIB} = -(\partial \mathcal{F}_{\rm IIB}/\partial T)$. Given the asymptotic behavior $G_9(x) \sim x^4 e^{-x}$, we see then that he entropy is then a monotonically decreasing increasing function T and a monotonically decreasing function in R. At fixed entropy of the universe, we see that the temperature T decreases with increasing R_9.

- $R_9 \ll T$: Using T-duality, we see that the temperature T now *increases* with increasing R_9 in the regime of small compact dimensions.

- $T^{-1} \gtrsim R_9 \gtrsim T$: The entropy is a function of temperature but is independent of R_9. Thus, the temperature is also independent of R_9.

All of this of course holds in the regime $T < T_H^{\rm IIB}$ (or equivalently $\beta > \beta_H^{\rm IIB}$), since this is the domain of validity of Eqn. (29). Putting everything together, we obtain a plot of temperature vs. compact dimension size as shown in Fig. 1. This figure was produced originally in [3], where an intuitive explanation for the shape was given. We have thus formalized the arguments presented in [3] using our expression for the free energy density of a Type IIB superstring gas.

A remarkable feature of Fig. 1 is the absence of a temperature singularity–the temperature remains well-behaved even as the universe grows smaller and smaller, due entirely to T-duality. As noted in [3], the string gas model therefore resolves an important issue plaguing the current early-universe paradigm.

VII. DISCUSSION AND CONCLUSION

Within the current paradigm of early-universe cosmology, the inflation theory suffers singularities in the regimes where quantum gravity is expected to be significant. Since string theory purports to be a theory of quantum gravity, it makes sense to consider a description of the early universe in terms of string theory. In particular, the string gas model does exactly this, considering the early universe as a high-temperature gas of ideal string states.

We considered the specific case of a Type IIB superstring gas, and we computed the corresponding one-loop partition function, following the approach in [6]. From this, we were able to calculate the Hagedorn temperature of the Type IIB superstring gas. We were also able to deduce a phase transition of such a gas in a universe with one compactified spatial dimension. Finally, we were able to formalize a qualitative understanding of the behavior of a Type IIB superstring gas as the spatial extent of the universe is decreased, following the analysis in [3]. As we describe, the temperature singularity problem in the classical inflationary theory is resolved by the string gas model.

Given the preliminary successes enjoyed by the string gas model, it is natural to consider several ways in which we might wish to extend this model to more completely describe early-universe cosmology. An important example is as follows: It can be shown [6] that the string gas free energy describes a radiation-dominated universe. However, there does not appear to be any clear mechanism from which matter can be generated from the string gas. Such a mechanism is indispensable in any model of early-universe cosmology, and the string gas theory would not be viable without it. Furthermore, the behavior of the string gas in the vicinity of the Hagedorn temperature is still poorly understood. It is imperative to understand this regime to develop a clearer picture of the phase structure of the string gas as a whole. We hope to address these major points in future work.

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