Lattice Mappings and Designs for Random Quantum Circuits

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In this survey paper, we discuss the statistical mechanical mapping for local random quantum circuits introduced by Nahum et al. [1] and Zhou et al. [2]. Following recent work by Jones [3], we also review the application of these methods to the convergence of these circuits to unitary k-designs, distributions that emulate the first moments of the Haar measure. This approach maps the quantum circuit to a triangular lattice of Ising-like spins in the symmetric group, whose configurations can be understood and tractable computed as energetically-unfavorable domain walls propagating and interacting through the lattice. This quite modern intersection of ideas in quantum information and condensed matter has found numerous recent applications to computing properties of these circuits, and here we detail their key ideas and raise their possible extensions while reviewing the necessary background in quantum information.

I. INTRODUCTION

Random quantum circuits (RQCs) are defined by local, nearest neighbor gates drawn from the Haar measure, which can be understood as the uniform distribution on unitary operators. Their study has found numerous recent applications in quantum information theory, quantum computing, and many-body physics. Their analytically tractable yet minimally structured dynamics have been used to model quantum information scrambling and chaotic evolution in complex strongly coupled systems [4, 5]. From a computational perspective, they generate efficient error correcting codes [6], approximately encrypt quantum states [7], and are the basis for the current experimental realizations of quantum supremacy experiments [8].

Put succinctly, much of the value of RQCs lies in efficiently generating quantum randomness. While a uniformly random unitary describing the full circuit evolution could require an exponential number of 2-qudit gates and classical bits, locally random unitaries on a fixed circuit architecture only require polynomially many [9]. An important question that arises is how fast these pseudo-random local circuits emulate the Haar measure. The key application we focus on in this review paper is that of a $k-$design [10], a random circuit that is indistinguishable from fully Haar uniform up to the first k moments. We formally introduce this concept in Section II C.

Although much of the original progress in studying these local circuits was through spectral methods [11–13], a recent statistical mechanical mapping has shown to prove tighter convergence bounds and generally broader applications. In Section II of this review paper, we dive into the statistical mechanical mapping of Nahum et al.[1] and Zhou et al.[2], by showing how expectations over locally random unitaries behave as an effective spin system over a lattice of directed triangular plaquettes. In sections III and IV, we survey the work of Jones [3] in applying this approach to study how fast locally random quantum circuits converge to approximate $k-$designs. In particular, in section III we study the convergence of the RQC to a 2-Design, through an analytical treatment of the combinatorics of the configurations of non-crossing domain walls in the lattice. In section IV, we generalize the domain wall treatment, and discuss the unique configurations that arise in their interaction. We then take the limit of a high qudit local dimension q, which is used to infer a conjectures [3] on the convergence for general k and q. Finally, in section V, we discuss recent general applications of these techniques and open problems.

II. THE LATTICE MAPPING

The lattice mapping for random quantum circuits (RQCs) to be discussed in this section was originally introduced by Nahum et al.[1] and Zhou et al.[2], in the context of computing out-of-time ordered correlation functions (OTOC) and entanglement entropies of local random quantum circuits. The model they studied was that of a 1D chain of n qudits in $\mathbb{C}^q$, which is evolved in time by t alternating layers of nearest neighbor 2-qudit Haar random gates in $U(q^2)$. We depict this alternating circuit geometry in the figure below.

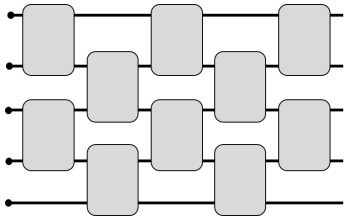

FIG. 1. 1D Random Quantum Circuit with 2-qudit gates

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A. The Weingarten Formula

In full generality, any polynomial expectation over the randomness of these gates can be written as an integral over the Haar measure of products of the entries of the unitaries, which we express below. Quite conveniently, however, is that these integrals have been extensively studied in the context of representation theory, where work by Collins and others [14, 15] has shown that these integrals behave as weighted index contractions between permutations.

$$\int dU \prod_{k}^{n} U_{i_k j_k} \times \prod_{k}^{m} U^{\dagger}_{i'_k j'_k} = \tag{1}$$

$$= \sum_{\sigma,\tau \in S_n} \mathcal{W}g(\sigma\tau^{-1}, q^2) \prod_{k} \delta_{i_k, j'_{\sigma(k)}} \times \prod_{k} \delta_{j_k, i'_{\tau(k)}} \tag{2}$$

Although the derivation of these results are a quite insightful discussion on Schur-Weyl Duality, for the purpose of this brief review we take it for granted. The weights here $\mathcal{W}g(\sigma\tau^{-1}, q^2)$ are the Weingarten functions, which have complex analytical expansions in terms of characters of the symmetric group S_k (the set of all permutations on k elements), but in general can be understood as the matrix inverse of the Gram matrix of permutation operators on the $k-$fold space.

For context, let us take a look at the Weingarten weights for $k = 2$. In this case, the symmetric group is composed of just the identity and swap permutations $S_2 = \{\mathbb{I}, F\}$. The entries of the gram matrix

$$\langle\sigma|G|\tau\rangle = \langle\Phi_\sigma|\Phi_\tau\rangle = \langle\Phi^{\otimes 2}|\mathbb{I}^{\otimes 2} \otimes P(\tau\sigma^{-1})|\Phi^{\otimes 2}\rangle = q^{-d(\sigma,\tau)} \tag{3}$$

which can generally be expressed in terms of the distance in transpositions $d(\sigma, \tau)$ between the two permutations, simplifies to the 2×2 matrix

$$G = \begin{bmatrix} 1 & q^{-1} \\ q^{-1} & 1 \end{bmatrix} \Rightarrow \mathcal{W}g = \frac{q^2}{q^2 - 1} \begin{bmatrix} 1 & -q^{-1} \\ -q^{-1} & 1 \end{bmatrix} \tag{4}$$

note that the off-diagonal entries are actually negative! Although the negative off-diagonal entries aren't a general property, they are a clear caveat in interpreting these expectations as partition functions. We'll return to this picture shortly.

B. Decimation

To compute physical quantities such as the entanglement entropies, the frame potential, and OTOC, we are generally interested in expectations over $V \otimes V^\dagger$ and its powers, where V here corresponds to the overall unitary describing the circuit evolution. The expectation of the kth moment $(V \otimes V^\dagger)^k$ of a locally RQC can be viewed as creating $2k$ copies of the circuit, with k copies of each gate U_{ij}, and k 'folded over' copies $U^*_{j'i'}$. What ensues is that the Weingarten formula enforces the contraction between incoming indices i, j' and outgoing indices j, i', defined by the permutations σ, τ, whereby the expectation over the copies of a given gate acts as an effective pair of vertices with spins σ and τ respectively. We represent this equivalence diagrammatically below, following the notation in a recent work by Choi et al. [16].

$$\mathbb{E}[(U \otimes U^*)^{\otimes k}] = \sum_{\sigma,\tau}$$

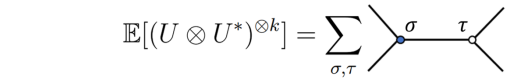


FIG. 2. Expectations as Effective Spins

These effective vertices with spins in the symmetric group define a hexagonal lattice on $n_l = \left\lfloor \frac{n}{2} \right\rfloor$ spins on the first layer and couplings on each edge, as pictured in figure 3. The horizontal edges denote couplings with the Weingarten weight $\mathcal{W}g(\sigma\tau^{-1}, q^2)$; while the diagonal lines indicate the index contraction with weight defined by the gram matrix G.

Unfortunately, as pointed out previously, some of these weights can be negative, thereby preventing any natural interpretation as a partition function. To resolve this issue, Nahum et al. [1] performed a decimation procedure by summing over all the intermediary (white) spins in the hexagonal lattice. This defines a triangular lattice, where all the couplings are positive. We depict this decimation procedure visually in figure 3.

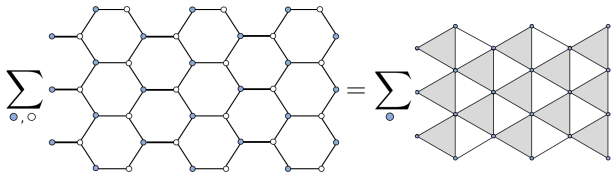
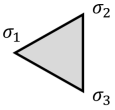


FIG. 3. The Lattices and the Decimation Procedure

In turn, this generates 3-spins couplings on directed triangular plaquettes, denoted in gray above. Their value, given 3 permutations $\sigma_1, \sigma_2, \sigma_3$, is defined by the sum

$$K^{\sigma_1}_{\sigma_2\sigma_3} = \sum_{\tau} \mathcal{W}g(\sigma_1\tau^{-1}) q^{-d(\sigma_2\tau)} q^{-d(\sigma_3\tau)} =$$



We can now finally write our partition function in terms of a summation over the spins in the permutation group, on the directed triangular lattice

$$Z = \sum_{\{\sigma\}} \prod_{\triangle} K^{\sigma_1}_{\sigma_2\sigma_3} \tag{5}$$

where the couplings are defined solely on the grey triangles of figure 3. However, the physical quantity this partition function represents is entirely defined by the boundary conditions we enforce. For example, the OTOC can be computed by fixing the boundary spins [1] and summing over the bulk, while general Renyi entropies [2] can be computed by partitioning the permutations on the right hand side to two regions of distinct uniform spins.

C. Boundary Conditions and the Frame Potential

In light of the applications we discuss in this review, in this subsection we turn to more technical definitions in quantum information, and the boundary conditions they enforce.

Formally, a k-design is an ensemble of unitaries S (a set with probability measure) that captures the first k moments of the Haar measure. Given an operator O acting on $(\mathbb{C}^q)^{\otimes k}$, we can define the k-fold channel

$$\Phi_S^k(O) = \int_S dU U^{\otimes k} O U^{\dagger \otimes k} \tag{6}$$

such that a $k-$design S satisfies $\Phi_S^k(O) = \Phi_{\text{Haar}}^k(O) \forall$ operators O. For $k = 1$, random Pauli's already serve as 1-designs, but not much more is known about exact designs for higher k. Instead, we could ask the question about how fast a circuit with limited randomness converges to a k-design, which in the framework of this section is now much more amenable to analytical treatment. An $\epsilon-$approximate k-design is defined to have error at most ϵ in diamond norm, i.e.,

$$\sup_m \sup_{O:|O|_1=1} \left| (\Phi_S^k \otimes \mathbb{I}_m - \Phi_{\text{Haar}}^k \otimes \mathbb{I}_m)(O) \right|_1 \leq \epsilon \tag{7}$$

and serves as a metric for the distinguishability of the channels. Although a direct computation of this distance is very complex, it has a straightforward upper bound in terms of the frame potential, $\mathcal{F}$ [3],

$$\leq q^{2nk}(\mathcal{F}_S^k - \mathcal{F}_{\text{Haar}}^k), \text{ where } \mathcal{F}_S^k = \int_{U,W \in S} |\text{Tr}[UW^\dagger]|^{2k} \tag{8}$$

which is a double average over the ensemble. The frame potential itself has quite a simply circuit interpretation. The double average $UW^\dagger$ simply acts as a longer circuit, with depth $2(t-1)$. When evaluating the kth moment, simply create $2k$ parallel copies to the circuit, where half of the gate copies are U, and the other half are the adjoint $U^\dagger$. Finally, the trace enforces time-periodic boundary conditions on the circuit, which translates to time-periodic boundary conditions on the lattice. That is, the spins on the left and right boundaries of the triangular lattice are fixed to be equal.

As in [3], the frame potential is lower bounded by its Haar value, i.e. $\mathcal{F}_S^k \geq \mathcal{F}_{\text{Haar}}^k$, which is known to be simply $\mathcal{F}_{\text{Haar}}^k = k!$. In the ensuing sections, we turn to computing this frame potential over the locally random quantum circuit ensemble ($S = RQC$) through the partition function method.

III. 2-DESIGNS AND THE NON-INTERACTING DOMAIN WALL APPROXIMATION

In the previous section, we reviewed the work of Zhou et al.[2] and Nahum et al.[1] in reducing the computation over local random unitaries to the combinatorial problem of computing a partition function on a directed triangular lattice of local S_k spins. In this section and the next, we turn to reviewing the work of [3] in treating these combinatorial problems as random walks of non-crossing and interacting domain walls. Here, we study the case $k = 2$, where we have Ising-like spins in the set $S_2 = \{\mathbb{I}, F\}$: the identity and swap permutations.

A. The Domain Wall Configurations

In this case, there are just $2^3 = 8$ relevant plaquette interactions, which we depict visually and quantify in figure 4 (up to exchange symmetry).

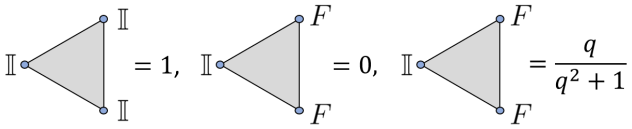


FIG. 4. Plaquette Interactions for $k = 2$ [3]

We note that there is a fascinating interpretation to the interactions above. In particular, that of energetically unfavorable domain walls separating uniform regions of $\mathbb{I}$ or F spins. Note how homogeneous plaquettes (the left-most in figure 4) have weight 1, while if two vertically adjacent permutations on a plaquette are equal (the middle plaquette in 4), then the third is forced to be the same (weight of 0). This behaviour in and of itself is quite similar to systems of frustrated spins.

More importantly, however, are the interactions in which $\mathbb{I}, F$ are adjacent in the vertical direction on a plaquette. These non-zero contributions correspond to a domain wall - a barrier between $\mathbb{I}$ or F spins, propagating in the time direction on the lattice. At each step, the domain wall moves either up or down, and contributes a multiplicative factor of $q/(q^2+1) < 1$ to the partition function, corresponding to an energy cost of having differing neighboring spins. The resulting partition function becomes

$$\mathcal{F}_{RQC}^2 = 2 + \sum \text{domain wall configurations} \tag{9}$$

we note, as discussed in Section II, that the valid configurations are a function of the boundary conditions enforced, which in turn are defined by the physical quantity we hope to compute. In the case of the frame potential, we have periodic boundary conditions in time: and so each domain wall must start and end at the same place. Moreover, domain walls can not intersect (cross), pair-create or annihilate, as these reduce simply to the weight 0 configurations in figure 4.

B. Non-Interacting Domain Walls and Tractibility

We proceed by expanding the partition function in terms of the number w of domain walls. Each configuration of w walls contributes a factor $q/(q^2+1)$ at each step of the depth $2(t-1)$ lattice, per wall. However, it becomes quite non-trivial to account for the non-crossing constraint of the walls. To do so, we instead upper bound the partition function - thereby upper bounding the distance from the RQC to the Haar Measure - by relaxing the overlapping constraint and the spatial boundaries, quite similar to the phantom loop approximation for 2D Ising Models. In this manner, each of w walls can start/end from in between the n_l left-boundary lattice points, and of its $2(t-1)$ steps, exactly $t-1$ must be upwards, and exactly $t-1$ must be downwards. We count these weighted configurations rather straightforwardly

$$\mathcal{F}^2_{RQC} \leq 2 + 2\sum_{w=1} \binom{n_l - 1}{w} \binom{2(t-1)}{t-1}^w \left(\frac{q}{q^2+1}\right)^{2w(t-1)} \tag{10}$$

$$\leq 2\left(1 + \binom{2(t-1)}{t-1}\left(\frac{q}{q^2+1}\right)^{2(t-1)}\right)^{n_l - 1} \approx 2(1+q^{-2t})^n \tag{11}$$

From Section II and after certain algebraic simplifications, we can now invert the ϵ distance in the diamond norm to find the depth t in which random circuits form an ϵ-approximate 2-design. What follows is summarized in the theorem

Theorem 1 [3] *Locally Random Quantum Circuits with 2-qudit gates form ϵ-approximate 2-designs in depth $t = O(n + \log 1/\epsilon)$.*

Although this approximation is quite crude, [2, 3] do point out that it is possible to perform an accurate expansion through the method of images, and account for boundaries and domain wall overlaps. However, these would only account for constant factors on the result above that already asymptotically matches the lower bounds previously mentioned in Section I.

IV. k-DESIGNS AND INTERACTING DOMAIN WALLS

Extending the framework of the previous section to arbitrary k faces several caveats. Although the lattice mapping of Section II remains precise, the plaquette terms now include general interactions $K^{\sigma_1}_{\sigma_2\sigma_3}$ between three permutations $\sigma_1, \sigma_2, \sigma_3 \in S_k$ which give rise to quite unique configurations. Although for the purposes of this survey we won't list the plaquette terms as before, it is instructive to discuss the non-trivial domain wall interactions that may arise. In the rest of this section, we follow Jones's [3] approach in studying the high-local dimension q expansion to the partition function and using it to motivate their conjecture on the fast convergence of k-designs, which remains an open problem in the study of random quantum circuits.

A. Diagrammatics and Domain Wall Creation

Following the approach of Zhou et al. [2], the types of plaquette terms that arise for general k can still be understood as energetically-unfavorable propagating domain walls between regions of uniform spin. If each spin corresponds to a permutation on k elements, then the domain walls are directly parametrized in terms of the transpositions (the 2 element swaps) that convert between the permutations on either side of the wall. For S_k, there are $\binom{k}{2}$ total transpositions, and any two permutations differ by at most $k-1$ of them. Jones [3] denotes this diagrammatically by representing the set of transpositions at each wall by distinct coloured lines, which we turn to discussing in this subsection.

First of all, exactly as in the $k=2$ case, homogeneous plaquettes have weight 1, and domain walls cannot spontaneously be annihilated. Put succintly, $K^{\sigma_1}_{\sigma_2\sigma_2} = \delta_{\sigma_1,\sigma_2}$, which can be represented as in figure 5.

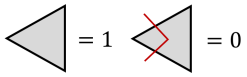


FIG. 5. Uniform plaquettes and disallowed domain wall annihlation [3]

More generally, we can have the intersection of 3 distinct domain walls at a plaquette that intersect, annihilate, and create new types of domain walls defined on the incoming transpositions. Take the top-rightmost diagram in figure 6 as an example. The two incoming spins define walls with two transpositions each, which annihlate the middle transposition to create a new domain wall with the two non-annilated transpositions. We represent other such higher order effects in the same figure.

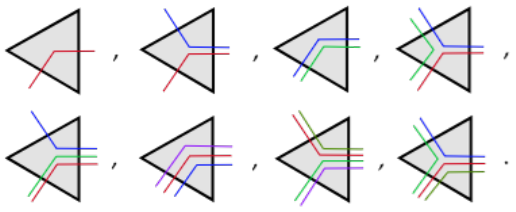


FIG. 6. Higher Order Plaquette Interactions

We note that interactions of this form permit the formation of closed loops of domain walls and generically unique configurations, which make the resulting computation quite challenging. These non-trivial effects are precisely why this problem has arisen as one of the fascinating questions in the study of random quantum circuits.

However, in the limit of high local dimension q considerable approximations can be made to show the decay of the Weingarten weight. As we discuss in the next section, to leading order in q^{-1}, bounding the partition function simply becomes a problem of analyzing the configurations of a single domain wall, without annihilation nor creation phenomena. In this manner, it becomes quite straightforward to analyze the convergence of the RQC to a k-design in the high q limit, which Jones [3] then uses to conjecture its behavior for general q.

B. The Local Dimension Expansion

In this subsection, we follow the approach of Jones [3] in taking the limit $q \to \infty$ and computing the leading order configurations to the partition function. They import the asymptotic expansion of the Weingarten function for large q, $\mathcal{W}g(\sigma\tau^{-1}, q^2) \sim q^{-2d(\sigma,\tau)}$ from the original work of Weingarten and recent work by Collins et al. [17, 18], such that the leading order plaquette coupling expansion becomes

$$K^{\sigma_1}_{\sigma_2\sigma_3} \approx \sum_{\tau \in S_k} q^{-2d(\sigma_1,\tau)} q^{-d(\sigma_2,\tau)} q^{-d(\sigma_3,\tau)} \tag{12}$$

$$\approx q^{-d(\sigma_1,\sigma_2)-d(\sigma_1,\sigma_3)} \tag{13}$$

here, the leading order contribution is at the maximum element of the sum, where $\tau = \sigma_1$, such that the exponent of q is simply the sum of the distance (in transpositions) of σ_1 to σ_2 and σ_3.

In this manner, the leading order contribution to the partition function is simply that of a single domain wall defined on a single transposition, propagating under time-periodic boundary conditions. The resulting combinatorics is quite similar to that of the single wall in the $k = 2$ case, with a multiplicative factor of $\binom{k}{2}$ in picking the defining transposition, and $k!$ in the ground state degeneracy. What arises is the expansion

$$\mathcal{F}^k_{RQC} = k!\left(1+\binom{k}{2}(n_l-1)\binom{2(t-1)}{t-1}\left(\frac{q}{q^2+1}\right)^{2(t-1)}+\cdots\right) \tag{14}$$

Now, once again we can invert the expression for the upper bound of the diamond distance in section II, to find the circuit depth where the local random quantum circuit converges to an ϵ-approximate k-design, up to leading order in q. We summarize in the following theorem,

Theorem 2 [3] *Locally Random Quantum Circuits with 2-qudit gates form ϵ-approximate k-designs in depth $t = O(kn \log q + k \log k + \log \frac{1}{\epsilon})$, up to leading order in the local dimension q.*

Jones [3] then ends his paper with the following conjecture. Configurations with multiple domain walls decay rapidly, and thereby single domain wall sectors of the lattice partition function *should* dominate the multidomain wall sectors for higher moments k and arbitrary local dimension q. In this manner, he postulates that locally random quantum circuits should converge to k designs within the assymptotic lower bound, in depth $O(nk)$.

V. DISCUSSION AND OUTLOOK

In this survey paper we discussed the statistical mechanical mapping of local random quantum circuits of Nahum et al. [1] and Zhou et al. [2], and followed Jones' [3] application of these techniques to approximate k-designs. We've reviewed some quite modern ideas at the intersection of quantum information and condensed matter physics which have recently found broad applications in the study of random quantum circuits, such as classical simulation [19], entanglement growth [20], and entanglement phase transitions [16, 21], and have left several fascinating open problems.

In particular, the extension of the mapping technique to general graph circuit architectures, random circuit models with symmetries, floquet random circuits, and different metrics of convergence to randomness, all remain largely unstudied.

In the context of $k-$designs, it would be fascinating to see applications of the lattice mapping to general problems such as the study of quantum pseudo-randomness, circuit compressibility, and quantum supremacy. As we discussed in Section IV, even the domain wall picture is incomplete for general k, and rigorous proofs of the convergence of k-designs within matching lower bounds have yet to be presented.

VI. ACKNOWLEDGEMENTS

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[1] A. Nahum, S. Vijay, and J. Haah, Operator spreading in random unitary circuits, Physical Review X **8**, 021014 (2017).

[2] T. Zhou and A. Nahum, Emergent statistical mechanics of entanglement in random unitary circuits, Physical Review B **99** (2018).

[3] N. Hunter-Jones, Unitary designs from statistical mechanics in random quantum circuits., arXiv: Quantum Physics (2019).

[4] P. Hayden and J. Preskill, Black holes as mirrors: Quantum information in random subsystems, Journal of High Energy Physics **2007**, 120 (2007).

[5] A. Nahum, J. Ruhman, S. Vijay, and J. Haah, Quantum entanglement growth under random unitary dynamics, arXiv: Statistical Mechanics (2016).

[6] W. Brown and O. Fawzi, Short random circuits define good quantum error correcting codes, 2013 IEEE International Symposium on Information Theory , 346 (2013).

[7] P. Hayden, D. Leung, P. Shor, and A. Winter, Randomizing quantum states: Constructions and applications, Communications in Mathematical Physics **250**, 371 (2004).

[8] S. Boixo, S. Isakov, V. Smelyanskiy, R. Babbush, N. Ding, Z. Jiang, M. Bremner, J. Martinis, and H. Neven, Characterizing quantum supremacy in near-term devices, Nature Physics **14**, 595 (2016).

[9] E. Knill, Approximation by quantum circuits, arXiv: Quantum Physics (1995).

[10] L. Arnaud and D. Braun, Efficiency of producing random unitary matrices with quantum circuits, Physical Review A **78** (2008).

[11] A. Harrow and R. A. Low, Random quantum circuits are approximate 2-designs, Communications in Mathematical Physics **291**, 257 (2009).

[12] F. Brandão, A. Harrow, and M. Horodecki, Local random quantum circuits are approximate polynomial-designs, Communications in Mathematical Physics **346**, 397 (2016).

[13] A. Dalzell, N. Hunter-Jones, and F. Brandão, Random quantum circuits anti-concentrate in log depth., arXiv: Quantum Physics (2020).

[14] B. Collins, 2 00 2 moments and cumulants of polynomial random variables on unitary groups , the itzykson zuber integral and free probability (2002).

[15] B. Collins and P. Śniady, Integration with respect to the haar measure on unitary, orthogonal and symplectic group, Communications in Mathematical Physics **264**, 773 (2006).

[16] S. Choi, Y. Bao, X.-L. Qi, and E. Altman, Quantum error correction in scrambling dynamics and measurement-induced phase transition., Physical review letters **125 3**, 030505 (2020).

[17] D. Weingarten, Asymptotic behavior of group integrals in the limit of infinite rank, Journal of Mathematical Physics **19**, 999 (1978).

[18] B. Collins and S. Matsumoto, Weingarten calculus via orthogonality relations: new applications, arXiv: Combinatorics (2017).

[19] J. Napp, R. L. Placa, A. Dalzell, F. Brandão, and A. Harrow, Efficient classical simulation of random shallow 2d quantum circuits, ArXiv **abs/2001.00021** (2020).

[20] F. Pollmann, T. Rakovszky, C. Keyserlingk, and S. Sondhi, Hydrodynamics in random unitary circuits with and without conservation laws, Bulletin of the American Physical Society **2018** (2018).

[21] C.-M. Jian, Y. You, R. Vasseur, and A. Ludwig, Measurement-induced criticality in random quantum circuits, Physical Review B **101**, 104302 (2020).