# 2D Ising Model on Hexgonal Lattices*

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Theories defined on discrete lattice provide a numerical method that complement the perturbation approach. We will review Ising model on a 2D hexagonal lattice, illustrate the critical exponents in the hexagonal lattice, and study the duality of this lattice. At the end of this paper, the deconfinement phase transition in 2D lattice gauge theory is also discussed.

## I. INTRODUCTION

Lattice models have played a critical role in statistical physics [1]. Some of these models are analytically solvable, and thus give an interpolation for the study beyond the physics from perturbation theory. If not solvable, the discreteness of the lattice models offers a system allowing numerical computation by using techniques such as Monte Carlo simulation. With the techniques such as renormalization group, duality transformation, etc., the lattice models are able to probe the phase transition properties.

In 2D Euclidean space, there are only five different Bravais lattices classified by their symmetry groups, including oblique, rectangular, centered rectangular, hexagonal(triangular), and square [2]. For each category, different translational orientations of the lattice will also affect further calculation. In the lecture, the upright square lattice (Migdal-Kadanoff approximation) and the triangular lattice (Niemeijer-van Leeuwen approximation) are discussed [1].

Though the critical exponents are universality, different choices of lattice structures require different level of approximation. We saw the magnetic and thermal exponents $(y_h, y_t)$ are different at linear approximation in different lattices. In this paper, I will treat 2D Ising model on a 2D hexagonal lattice.

## II. ISING MODEL ON HEXAGONAL LATTICES

In a square lattice, new interactions are generated in renormalization group (RG), and thus the nearest neighbor interaction is not preserved. To avoid the increase of new interactions in each RG step, consider Ising model on a hexagonal lattice with nearest neighbor Hamiltonian $-\beta\mathcal{H} = K\sum_{<i,j>}\sigma_i\sigma_j$, $\sigma_i = \pm 1$. We will show a position space renormalization group with length scale factor $b = 2$.

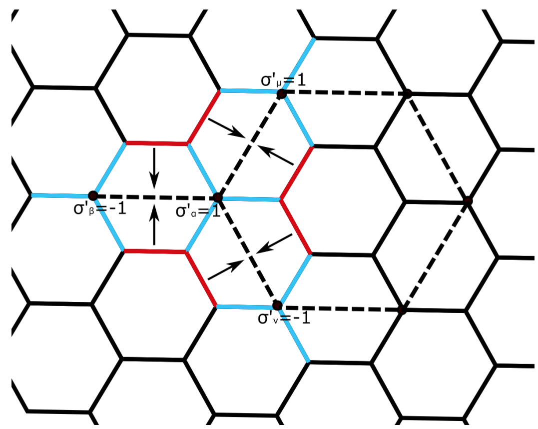


FIG. 1. The intracell interactions are labeled by blue, while the intercell interactions are labeled by red. Black dash line stands for the nearest-neighbor interaction in cluster-spin system.

### A. Mapping on A Hexagonal Lattice

The transformation $\{\sigma_i\} \to \{\sigma'_i\}$ reduce the degrees of freedom by the factor $b$. In the hexagon lattice, the original lattice sites are grouped into clusters of four, as shown in Fig. 1.

In this mapping, the majority-rule is applied with modification that more weight $1+\epsilon$ is on central site spin $\sigma_0$, which breaks the ill-defined case when $\sum_{i=0}^{3}\sigma_i = 0$.

$$\sigma'_0 = sign[(1+\epsilon)\sigma_0 + \sigma_1 + \sigma_2 + \sigma_3]. \tag{1}$$

The renormalized interaction corresponding to the mapping above is obtained from the constrained sum,

$$e^{-\beta\mathcal{H}'[\sigma'_i]} = \sum_{\{\sigma_i\}\to\{\sigma'_i\}}^{\prime} e^{-\beta\mathcal{H}[\sigma_i]}. \tag{2}$$

Now, each cluster-spin contains 4 site-spins and 3 interactions. These interactions inside a cluster spin is defined as intracell interactions. With all possible configurations $\{\sigma_i\}$, the contribution from intracell interactions can be $3K(\sigma_0 = \sigma_{1,2,3})$, $K(\sigma_0 = \sigma_{1,2} \neq \sigma_3)$, $-K(\sigma_0 = \sigma_1 \neq \sigma_{2,3})$, and $-3K(\sigma_0 \neq \sigma_{1,2,3})$.

The contribution outside the cluster-spin is defined as intercell interactions. This interaction between $\sigma'_\alpha$ and $\sigma'_\beta$ contains two bonds, so the possible values of intercell

* This paper is inspired by the renormalization group problem in Chap.6 of *Statistical Physics of Fields* [1].

interaction can be $2K$(2 bonds are positive), $0$(2 bonds are opposite), and $-2K$(2 bonds are negative).

### B. Renormalization Group

There are four possible configurations of the bond in cluster-spin system, $\sigma'_{i,j} = (+,+), (-,-), (+,-), (-,+)$, while the last two are the same. For a given value of a cluster–spin, various possible configurations of site–spins exist. A cluster-spin interaction should contain all possible added-up values of the intracell interactions and the intercell interactions.

The general interaction between two spins also contains an energy shift $g$, which has no effect on the critical properties. With the renormalized interaction $\mathcal{R}(\sigma'_i, \sigma'_j)$, the parallel and anti-parallel clusters are given by

$$e^{K'+g'} \equiv \mathcal{R}(+,+) = e^{8K} + 2e^{6K} + 7e^{4K} + 14e^{2K} + 17 + 14e^{-2K} + 7e^{-4K} + 2e^{-6K},$$
$$e^{-K'+g'} \equiv \mathcal{R}(+,-) = 9e^{4K} + 16e^{2K} + 13 + 16e^{-2K} + 9e^{-4K} + e^{-8K}.$$

Solve these equations, we get the resulting recursion relation for the nearest-neighbor interactions $K'$,

$$K' = \frac{1}{2}\ln[\frac{\mathcal{R}(+,+)}{\mathcal{R}(+,-)}]. \tag{3}$$

If $K \ll 1$, $K' \approx \frac{1}{2}\ln[1+\frac{K}{4}] \approx \frac{K}{4} < K$, the high-temperature sink at $K^* = 0$ thus has a stable fixed point with zero correlation length. And if $K \gg 1$, $K' \approx \frac{1}{2}\ln[\frac{e^{4K}}{9}] \approx 2K > K$, so the low temperature fixed point at $K^* \to \infty$ is also stable.

There must be an unstable fixed point at finite $K' = K$, the critical ferromagnetic coupling is obtained $K^* = 1.045$. From this RG scheme, $y_t = \frac{1}{\ln(b)}\frac{\partial K'}{\partial K}\Big|_{K^*} = 0.862$. We can estimate the exponent $\nu = \frac{1}{y_t} = 1.16$, very close to the exact value $\nu = 1$.

## III. DUALITY IN HEXAGONAL LATTICES

In the two theories that are dual to each other, if one has a strong interaction, the other has a weak interaction. In some problems where duality can be described more accurately, we can even build a one-to-one correspondence to combine the variables in one side of the theory with other variables on the other side. This is a characteristic of almost all duality.

However, we should not directly compare the hexagonal lattice with its dual theory, since the geometrical dual of a hexagonal lattice is a triangular lattice. To have the self-dual, we will regroup the hexagonal lattice to triangular lattice, and then get the dual hexagonal lattice from that.

### A. Regroup to Triangular Lattice

A hexagonal lattice can be considered as the combination of two triangular sub-lattices. To form a new lattice, we remove one sub-lattice by summing over the its sites [3]. Take Fig. 2(a) as an example, the site $\sigma_0$ is removed and new nearest-neighbor interaction for the triangular lattice, $K_t$, is defined,

$$\sum_{\sigma_0=\pm1} e^{K\sigma_0(\sigma_1+\sigma_2+\sigma_3)} = e^{g_t+K_t(\sigma_1\sigma_2+\sigma_2\sigma_3+\sigma_3\sigma_1)},$$

where $g_t$ energy offset in the triangular lattice.

Two possible configurations of the site-spins. $(\sigma_1, \sigma_2, \sigma_3) = (1,1,1)$ and $(1,1,-1)$ give the equations $e^{3K} + e^{-3K} = e^{3K_t+g_t}$, and $e^K + e^{-K} = e^{-K_t+g_t}$ respectively. The relation between couplings $K$ and $K_t$ is,

$$e^{4K_t} = \frac{e^{3K}+e^{-3K}}{e^K+e^{-K}} = \frac{\cosh 3K}{\cosh K}. \tag{4}$$

Remark this transformation reduces the degrees of freedom by half, from $2N$ to $N$. However, the number of bonds $N_b = 3N$ is not reduced, since each site is

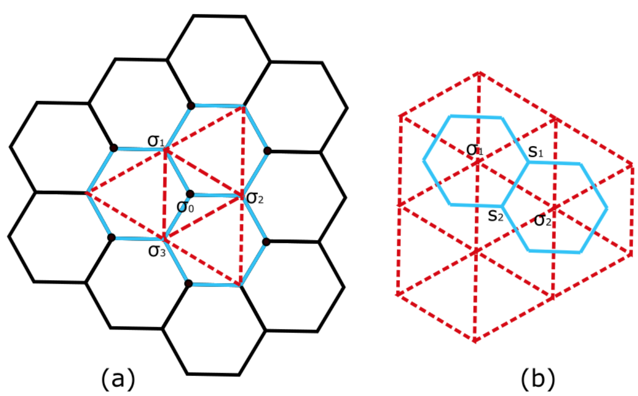


FIG. 2. In both figures, the dash red bonds are for triangular interactions and the solid blue bonds are for hexagonal interactions. However, in figure (a) the hexagonal lattice is the original theory, while in figure (b) the hexagonal lattice is the dual theory.

connected to twice as many neighboring sites in the triangular lattice. There are now $2N$ constraints associated with the plaquettes of this lattice.

### B. Critical Couplings

The dual system of triangular lattice is shown in Fig. 2(b). The mapping of duality is straightforward, each bond of the dual lattice is perpendicular to a bond of the original lattice.

Insert constraints into the triangular partition function of bonds $b_{ij} = \sigma_i \sigma_j$, Notice each bond is associated with two plaquettes $s_i$, $s_j$, where $s_p$ shows whether this plaquette $p$ is occupied or not.

$$\begin{aligned} Z \propto & \sum_{\{s_i\}} \sum_{\{b_{ij}\}} \prod_{\langle i,j \rangle} e^{K_t b_{ij}} b_{ij}^{s_i+s_j} \\ \propto & \sum_{\{s_i\}} \prod_{\langle i,j \rangle} e^{K_t} + e^{-K_t} (-1)^{s_i+s_j}. \end{aligned} \tag{5}$$

For the dual hexagonal partition function, use $\tilde{K}$ and $\tilde{g}$ as nearest-neighbor coupling and energy offset respectively,

$$Z \propto \sum_{\{s_i\}} \prod_{\langle i,j \rangle} e^{\tilde{K} s_i s_j + \tilde{g}}. \tag{6}$$

Match up the two partition functions, the dual relation is $e^{K_t} + e^{-K_t}(-1)^{s_i+s_j} = e^{\tilde{K} s_i s_j + \tilde{g}}$. And use the parallel and anti-parallel configurations of $s_{i,j}$, we get $e^{K_t} + e^{-K_t} = e^{\tilde{K}+\tilde{g}}$ and $e^{K_t} - e^{-K_t} = e^{-\tilde{K}+\tilde{g}}$. Solve these equations, the relation between couplings $K_t$ and $\tilde{K}$ is,

$$e^{2\tilde{K}} = \frac{e^{K_t} + e^{-K_t}}{e^{K_t} - e^{-K_t}} = \coth K_t. \tag{7}$$

Comparing equations (4) and (7) indicates the relation between the dual coupling and the original coupling,

$$\tilde{K}(K) = \frac{1}{2} \ln [\frac{\sqrt{\cosh 3K} + \sqrt{\cosh K}}{\sqrt{\cosh 3K} - \sqrt{\cosh K}}]. \tag{8}$$

It is not obvious from the function above, but when $\tilde{K}$ is monotonic decreasing with $K$. For the self-dual $K = \tilde{K}$, the critical coupling is obtained $K^* = \ln[\sqrt{2+\sqrt{3}}] = 0.658$.

## IV. 2D CONFINEMENT

The lattice models can also be used to study the phase of confinement, such as in quantum chromodynamics (QCD) [4]. To describe the fermionic theory, we apply the Grassmann algebra [5], which is omitted here.

In the confined phase, the energy between two particles are increasing with their separation distance, while in deconfined phase, it is a constant. To study the phase transition between confinement and deconfinement, the Wilson loop is an important observable. It is an order parameter for the deconfinement transition at finite temperature.

When static potential $V(r)$ grows indefinitely with the separation distance, $\langle W \rangle$ has to vanish [6]. And when $V(r)$ is a constant, so is $\langle W \rangle$. Thus we have,

$$\begin{aligned} &\lim_{r\to\infty} V(r) \propto r \ \Leftrightarrow \langle W \rangle = 0 \Rightarrow confinement, \\ &\lim_{r\to\infty} V(r) = C \Leftrightarrow \langle W \rangle \neq 0 \Rightarrow deconfinement. \end{aligned}$$

Start with the partition function

$$\begin{aligned} \tilde{Z} = & \sum_{\{s_i\}} e^{K \sum_{\langle i,j \rangle} s_i s_j} \\ \propto & \sum_{\{U_P\}} e^{J \sum_P [U_P + U_P^\dagger - \lambda (U_P + U_P^\dagger)^2]}, \end{aligned} \tag{9}$$

where $U_P$ and $U_P^\dagger$ are the plaquettes with opposite directions. The model is considered with a plaquette coupling J and a Rokhsar-Kivelson coupling $\lambda$ [7]. Here we will simply use $\lambda = 0$.

The Wilson loop is constructed by link variables contained in the contour $\mathcal{C}$ of the closed path,

$$\langle W \rangle = \langle \prod_{l \in \mathcal{C}} U_l \rangle = \frac{1}{\tilde{Z}} \sum_{\{U_P\}} \prod_{l \in \mathcal{C}} U_l e^{J \sum_P (U_P + U_P^\dagger)}. \tag{10}$$

For small $J$, use the expansion $e^{J \sum_P (U_P + U_P^\dagger)} = \sum_{i,j=0}^{\infty} \frac{J^{i+j}}{i!j!} (\sum_P U_P)^i (\sum_P U_P^\dagger)^j$, and note that the plaquettes used to fill the contour should have the opposite orientations of the Wilson loop,

$$\begin{aligned} \langle W \rangle = & \sum_{\{U_P\}} \frac{1}{S_A!} J^{S_A} (\sum_P U_P^\dagger)^{S_A} \prod_{l \in \mathcal{C}} U_l \\ \propto & \ e^{S_A \ln(J)}, \end{aligned} \tag{11}$$

where $S_A$ is the area of $A$. Regarding one dimension of the lattice as time, the static potential is given as $V(r) = -\ln(J)\ r$ [8]. This is a string tension that increases with the separation distance.

For weak coupling (large $J$), consider the ground state would be all plaquettes are in the same orientation, say $U_P^\dagger$. Excitations will flip some plaquettes to the opposite orientation. This also cause an extra $-1$ factor if the plaquettes are at the boundary of the loop, due to the link varibales $U_l$. With $N_P$ for the number of total plaquettes and $N_W$ for the number of plaquettes on the boundary of the loop, the Wilson loop is

$$\begin{aligned} \langle W \rangle = & \frac{1 + (N_P - N_W) e^{-2J} - N_W e^{-2J} + \cdots}{1 + N_P e^{-2J} + \cdots} \\ \approx & \ e^{-N_W 2 e^{-2J}}. \end{aligned} \tag{12}$$

In this case, as time goes to infinity, the static potential can be considered as a constant. This implies the deconfined phase.

We can conclude that under the $U(1)$ lattice gauge theory, it has a deconfinement phase transition at finite temperature. This result is the same as in the 3D Ising model under $Z_2$ lattice gauge theory shown in the book [1]. Without using any setups for the hexagonal lattice, this phase transition is universal.

## V. CONCLUSION

In this report we show the characteristics of Ising model on hexagonal lattice. The critical exponents on hexagonal lattice are different from that in the square lattice, which is caused by the limitation of approximation level. On the other hand, the phase transition is universal, that the hexagonal lattice does not play any unique role here.


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