

# The Theory of Planar Ising Models

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The square lattice Ising model is a famous model from statistical mechanics that both has a nontrivial phase transition and can be exactly solved. In this paper, we provide a self-contained derivation of the Kac-Ward combinatorial solution to general planar Ising models, discuss general features shared by all planar models such as their solvability in polynomial time, frustration, and dualities, and then close by computing the exact partition functions and critical temperatures of the triangular and hexagonal lattice Ising models.

## I. INTRODUCTION

The Ising model is a simple model of magnetism, where two-valued spins  $\sigma_i = \pm 1$  live on the vertices of a graph and interact through the graph's edges with energy  $-J_{ij}\sigma_i\sigma_j$ . Although it is impossible to compute the partition function of arbitrary Ising models due to the exponentially large number of configurations, certain special cases are exactly solvable. In particular, for planar Ising models (embeddable in the plane without self-intersection), Kac and Ward developed a way to sum the high-temperature expansion exactly and express the partition function as the determinant of a matrix with dimension equal to twice the number of bonds in the model. With the help of the exact solution, we examine a number of striking properties of Ising models, such as phase transitions, frustration, and duality.

## II. COMBINATORIAL SOLUTION

Our solution for general planar models largely follows the solution of the square lattice presented by M. Kardar in his graduate text, but we take additional care to handle vertices of degree exceeding four and to work with nonuniform couplings. The partition function is:

$$Z = \sum_{\{\sigma_i\}} \exp\left(\beta \sum_{\langle i,j \rangle} J_{ij}\sigma_i\sigma_j\right) = \sum_{\{\sigma_i\}} \prod_{\langle i,j \rangle} \exp(\beta J_{ij}\sigma_i\sigma_j), \quad (1)$$

where the spins  $\sigma_i$  take values  $\pm 1$ ,  $\langle i,j \rangle$  denotes summation over all of the model's bonds, and  $J_{ij}$  is bond  $ij$ 's coupling strength. Additionally, let  $N_s$  and  $N_b$  denote the number of spins and bonds in the model respectively. Because  $\sigma_i\sigma_j$  only takes values  $\pm 1$ , we have the identity:

$$\begin{aligned} \exp(\beta J_{ij}\sigma_i\sigma_j) &= \cosh \beta J_{ij} + \sigma_i\sigma_j \sinh \beta J_{ij} \\ &= \cosh \beta J_{ij}(1 + \sigma_i\sigma_j \tanh \beta J_{ij}). \end{aligned} \quad (2)$$

Substituting Eq. 2 into Eq. 1 yields:

$$Z = \left(\prod_{\langle i,j \rangle} \cosh \beta J_{ij}\right) \sum_{\{\sigma_i\}} \prod_{\langle i,j \rangle} (1 + \sigma_i\sigma_j \tanh \beta J_{ij}). \quad (3)$$

The hyperbolic cosine prefactor is analytical even in the thermodynamic limit, so any phase transition must

be encoded in the second product with the hyperbolic tangents. To calculate this, we can perform a high-temperature expansion in powers of  $\tanh \beta J_{ij} \equiv t_{ij}$ . In expanding the product over bonds of  $(1 + \sigma_i\sigma_j t_{ij})$ , each bond can either contribute 1 or  $t_{ij}$ , yielding a total of  $2^{N_b}$  terms. We represent these terms diagrammatically by drawing the Ising model as a graph where the spins are vertices and the bonds are edges and highlighting the bonds that contribute  $t_{ij}$ . If a spin  $\sigma_i$  has  $n_i$  bonds highlighted, then the term will have a factor of  $\sigma_i^{n_i}$ . After summing over  $\sigma_i = \pm 1$ ,  $\sigma_i^{n_i}$  becomes 2 if  $n_i$  is even or 0 if  $n_i$  is odd. Therefore, the only nonzero terms are those where each vertex has an even number of bonds highlighted, which are represented by closed diagrams. The partition function is:

$$\begin{aligned} Z &= Z_0 \sum_{\text{closed } G} \prod_{\langle i,j \rangle \in G} t_{ij} \\ Z_0 &= 2^{N_s} \prod_{\langle i,j \rangle} \cosh \beta J_{ij}. \end{aligned} \quad (4)$$

Because closed diagrams look like combinations of loops, one may naively factorize  $Z/Z_0$  as the exponential of all one-loop diagrams. However, there are ambiguities in interpreting how many loops a diagram is composed of, and some loops or combinations of loops do not form legitimate closed diagrams, leading to overcounting in the naive factorization. These all arise from cases where a vertex has more than two bonds highlighted. For example, the three ways to interpret a figure-eight diagram as a combination of loops are shown in Fig. 1. This issue can be corrected by assigning a negative or positive sign to all possible ways of splitting a diagram into a combination of loops and then summing over all splittings. For



FIG. 1: The figure eight diagram can be split in three ways: as a single loop without self-intersection, as a single loop that intersects itself, and as two non-intersecting loops. By assigning a factor of  $(-1)$  to the crossing in the second case and summing over splittings, all but one diagram is cancelled, correcting the overcounting.

a particular splitting, we assign a factor of  $(-1)$  to every intersection of two lines, whether the lines belong to the same loop or not. Note that two lines touching at a corner, such as in the third splitting drawn in Fig. 1, does not count as an intersection. To prove that this corrects the overcounting, it is sufficient to show inductively that our scheme corrects the overcounting at a single vertex. Consider a vertex in our closed graph with  $2n$  highlighted bonds. The base case is  $n = 2$ , where our analysis of the figure-eight diagram demonstrates that our scheme is correct. There are three splittings, two of them do not get a sign, and one does. This yields the correct total of 1, which is consistent with the fact that there is only one closed graph.

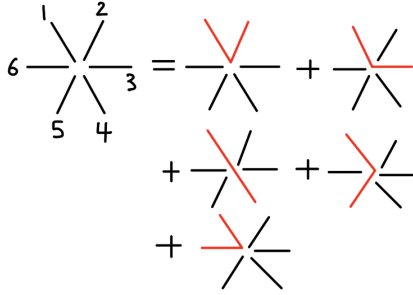


FIG. 2: Splitting a vertex with  $2n$  bonds is equivalent to grouping the bonds into  $n$  pairs. Our proof that our scheme is correct uses casework; once we choose the first two bonds to pair, drawn in red above, the problem is reduced to pairing the remaining  $2(n - 1)$  bonds. The figure above is an example for  $n = 3$ , where there are  $2n - 1 = 5$  cases to consider.

For the inductive step, consider a vertex with  $2n > 4$  bonds as drawn in Fig. 2, and label the bonds clockwise as  $\{1, 2, \dots, 2n\}$ . Splitting this vertex is equivalent to grouping the bonds into  $n$  pairs. Let the sum over all splittings, with each intersection contributing  $(-1)$ , be  $S(n)$ . We do casework on how bond 1 is paired. If we pair 1 with  $k \in \{2, 3, \dots, 2n\}$ , then we must pair  $\{2, 3, \dots, 2n\}/k$  amongst themselves. For a particular choice of pairing, the total number of intersections is equal to the number of intersections among  $\{2, 3, \dots, 2n\}/k$ , plus the number of intersections contributed by 1 and  $k$ . Bonds 1 and  $k$  form a wall between bonds  $\{2, 3, \dots, k - 1\}$  and bonds  $\{k + 1, \dots, 2n\}$ , so if bonds  $\{2, 3, \dots, k - 1\}$  form  $p$  pairs amongst themselves, the remaining  $k - 2 - 2p$  bonds must cross the  $1k$  line to pair with bonds  $\{k + 1, \dots, 2n\}$ . However,  $k - 2 - 2p$  has the same parity as  $k$ , so  $(-1)^{k-2-2p} = (-1)^k$  does not depend on  $p$ , and the sum of all splittings where 1 pairs with  $k$  is just  $(-1)^k S(n - 1) = (-1)^k$ . Summing over all  $k$  yields  $S(n) = \sum_{k=2}^{2n} (-1)^k = (n) - (n - 1) = 1$ . Because  $S(1) = 1$ ,  $S(n) = 1$  for all  $n$ , and the overcounting is corrected. In Fig. 3, we have explicitly drawn all 15 ways to split a six-bond vertex to more concretely demonstrate the overcounting correction.

There are two more potential issues with factorizing the sum of all closed diagrams into the exponential of

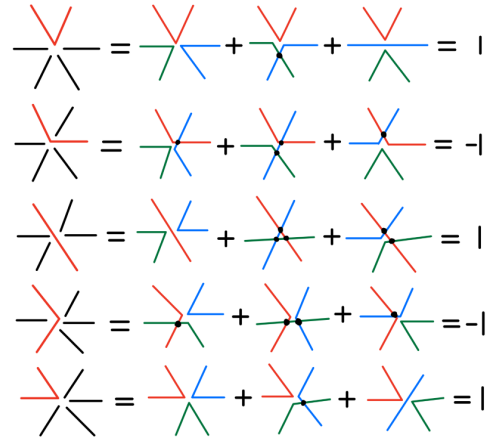


FIG. 3: All  $15 = 5 \cdot 3$  ways to split a vertex with six bonds into loops, with connected bonds drawn in the same color and intersections marked by a black dot. We can explicitly see that assigning  $(-1)$  to each intersection produces the correct total of 1 after all splittings are summed over.

one-loop diagrams. In forming a single loop or joining two or more loops, it is possible to create a situation in which the same bond is traversed twice. This creates vertices with an odd number of bonds and a final diagram that is not closed. Assigning  $(-1)$  per intersection actually fixes this issue, as long as we exclude loops with U-turns. When a bond is traversed by two walkers (we could imagine that a little ant is walking on the Ising model and marking its path), the two walkers can traverse the bond without crossing to make an “equals-sign”, or they can traverse the bond and switch to form a “cross”. These two cases differ by a single intersection, so they cancel, eliminating the invalid diagram. This fails only for U-turns, where the two cases are exactly the same walk, so they must be explicitly avoided. Avoiding U-turns is a local constraint and hence easy to implement, so this is not a big deal. Finally, in constructing

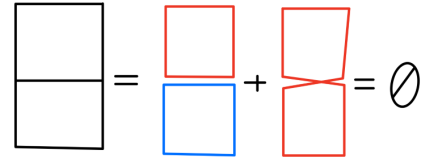


FIG. 4: Because it has vertices with three bonds, the black diagram is not closed. Such problematic bonds can be formed either by the touching of two loops or by a single loop returning to an already-traversed bond, but the  $(-1)$  per intersection cancels such diagrams.

closed diagrams by combining multiple loops, we need to account for intersections between different loops, which at first glance is an “interaction” between loops and spoils our factorization. Luckily, mathematicians have proved the intuitively true fact that two loops in the plane can only intersect an even number of times, so intersections between different loops wash out. The only exception

to this statement is a situation where two loops are tangent or meet at a corner, but we don't count these as intersections anyways.

Therefore, with all the potential sources of overcounting corrected for, we can write:

$$\sum_{\text{closed } G'} \prod_{\langle i,j \rangle \in G'} t_{ij} = \exp \left[ \sum_{\text{loops } G'} (-1)^{n_c} \prod_{\langle i,j \rangle \in G'} t_{ij} \right], \quad (5)$$

where the prime in  $G'$  reminds us that  $G'$  should not contain U-turns, and  $n_c$  is the number of self-intersections that the loop has. With this, the partition function becomes:

$$\ln Z = \ln Z_0 + \sum_{\text{loops } G'} (-1)^{n_c} \prod_{\langle i,j \rangle \in G'} t_{ij}. \quad (6)$$

All we need is a way to sum over U-turn-free loops with their respective signs and factors of  $t_{ij}$ . In two dimensions only, we can use Whitney's theorem to compute  $(-1)^{n_c}$  locally by integrating the loop's turn angle:

$$(-1)^{n_c} = -e^{i\theta/2}, \quad (7)$$

where  $\theta$  is the total angle that the tangent vector of the loop winds through. With this, all the loops  $G'$  contributing to the partition function and their phases can be efficiently computed by viewing the loops as Markovian random walks. The typical derivation of the square-lattice Ising model considers a walker hopping from vertex to vertex and keeps track of the walker's direction to compute the turn angle. It is slightly easier to instead regard the walker as living on the directed edges of the Ising model and hopping from edge to edge through the vertices. Let the pair  $ji$  denote the directed edge from vertex  $i$  to vertex  $j$ , and assign to it the orthonormal unit vector  $|ji\rangle$ . Additionally, let  $\theta_{ji}$  be the heading of directed edge  $ji$  with respect to some reference, such as the horizontal. For all allowed hops  $ji \rightarrow lk$ , we define the hopping matrix  $W(\beta)$  through its matrix element (all other matrix elements of  $W(\beta)$  are zero):

$$\langle lk|W(\beta)|ji\rangle = \exp\left(\frac{i}{2}(\theta_{lk} - \theta_{ji})\right) \sqrt{t_{lk}t_{ji}}, \quad (8)$$

where the square root needs to be taken with a grain of salt due to its multivalued nature. We take the root with argument 0 (positive real) when both  $t_{lk}$  and  $t_{ji}$  are positive, the root with argument  $\pi/2$  (positive imaginary) when one is positive and one is negative, and the root with argument  $\pi$  (negative real) when both are negative. The square root gives the correct factors of  $t$  in a symmetric way, and the complex exponential records the winding of the tangent vector. With this matrix, the sum of all closed loops of  $l$  steps including the coupling-strength factors is easily expressed as  $-\frac{1}{2l} \text{Tr } W(\beta)^l$ . The factor of 2 in the denominator cancels overcounting from considering the loops as oriented, and the factor of  $l$  cancels overcounting from forming the same loop by starting in multiple places. Summing over loops of all possible

lengths, applying the power series expansion of the logarithm, and using the matrix identity  $\text{Tr } \ln M = \ln \det M$  yields the final expression:

$$\ln Z = \ln Z_0 + \frac{1}{2} \ln \det (1 - W(\beta)). \quad (9)$$

### III. HOPPING EIGENVALUES

This compact expression gives the partition function of any planar Ising model, and its derivatives generate all bulk thermodynamic quantities. We can already see a number of features that all planar Ising models share. We know from graph theory that a planar graph of  $N_s$  vertices has less than  $3N_s$  undirected edges, or in other words planar graphs have average degree strictly less than 6. This is intuitively true, as the triangular lattice has coordination number 6 and is the most efficient way to pack circles in the plane. Furthermore, planarity strongly restricts the Ising model from being highly connected, meaning that  $1 - W(\beta)$  is very sparse and on average has only a couple of nonzero entries per row and column. For arbitrary matrices of dimension  $N$ , the determinant's complexity is  $O(N^3)$  using Gaussian elimination, although one can do significantly better than this in our case by exploiting  $1 - W(\beta)$ 's sparsity through techniques such as sparse LU factorization. From this, we conclude the remarkable result that the partition function and all other bulk thermodynamic quantities of planar Ising models have complexity at most  $O(N_s^3)$ .

We can also express the partition function in terms of  $W(\beta)$ 's eigenvalues. From the fact that the determinant is equal to the product of the eigenvalues, we have:

$$\begin{aligned} \ln Z &= \ln Z_0 + \frac{1}{2} \sum_{\lambda} \ln (1 - \lambda(\beta)), \\ Z &= Z_0 \sqrt{\prod_{\lambda} (1 - \lambda(\beta))}, \end{aligned} \quad (10)$$

where the eigenvalues  $\lambda(\beta)$  are implicitly functions of  $\beta$ . Although it is clearly not Hermitian, the hopping matrix does have an interesting symmetry:

$$\langle lk|W(\beta)|ji\rangle = \langle ij|W(\beta)|kl\rangle^*. \quad (11)$$

This is because if the walker can hop from  $|ji\rangle$  to  $|lk\rangle$ , then the walker must also be able to hop from  $|kl\rangle$  to  $|ij\rangle$ . Everything about these two hops is identical, except for the changes in turn angle which are opposites, so the matrix elements are related by complex conjugation. Define the matrix  $S$  through its action on the basis  $S|ij\rangle = |ji\rangle$ . By inspection,  $S^2 = I$ , and  $S$  is diagonal in the basis  $|ij\rangle_{\pm} = \frac{1}{\sqrt{2}}(|ij\rangle \pm |ji\rangle)$  with eigenvalues  $S|ij\rangle_{\pm} = \pm 1|ij\rangle_{\pm}$ . Therefore,  $S$  is real, symmetric, and idempotent. Our previous identity can be expressed us-

ing  $S$  as:

$$\begin{aligned} \langle lk|W(\beta)|ji\rangle &= \langle ij|W(\beta)^*|kl\rangle \\ &= \langle kl|W(\beta)^\dagger|ij\rangle \\ &= \langle lk|SW(\beta)^\dagger S|ji\rangle. \end{aligned} \quad (12)$$

Since  $S^{-1} = S$ , this implies that  $W(\beta) = S^{-1}W(\beta)^\dagger S$ . Thus,  $W(\beta)$  has the same eigenvalues as  $S^{-1}W(\beta)^\dagger S$ , which has the same eigenvalues as  $W(\beta)^\dagger$  since similarity transforms do not change a matrix's spectrum. Because  $W(\beta)^\dagger$ 's spectrum is the complex conjugate of  $W(\beta)$ 's spectrum,  $W(\beta)$ 's eigenvalues must come in complex conjugate pairs. Because  $\ln z^* = (\ln z)^*$  except for negative real  $z$ , the fact that the eigenvalues come in complex conjugate pairs guarantees that the free energy is real, as it should be.

Another interesting question to consider is whether an Ising model will have the same partition function when all its couplings are flipped in sign through  $J_{ij} \rightarrow -J_{ij}$ , which can just be thought of as keeping all the couplings the same but making the coldness negative. This is relevant to determining whether a non-frustrated antiferromagnetic state can exist in a model or not. In general, the closed graphs of planar Ising models are unions of plaquettes. Suppose that all plaquettes have an even number of edges, and consider constructing a closed diagram recursively by repeatedly merging it with additional plaquettes. At the first step in the recursion, we start with a single plaquette, which by assumption has an even number of edges. Suppose we add a plaquette with  $p$  edges to the existing graph. We gain  $p - 2s$  edges, where  $s$  is the number of edges that the new plaquette shares with the existing graph.  $p - 2s$  is even since  $p$  is even, so by induction all closed graphs have an even number of edges. Therefore, because each edge picks up a sign under  $\beta \rightarrow -\beta$ , but there are always an even number of edges in a closed diagram,  $\ln Z$  is a symmetric function of  $\beta$ . Since the zeros of  $\ln Z$  are defined through the hopping eigenvalues, this actually allows us to conclude that for Ising models made of plaquettes with an even number of edges, if  $\lambda$  is an eigenvalue so is  $-\lambda$ , granting us an additional symmetry. The converse is also true: if an Ising model has even a single plaquette with an odd number of edges, the  $\ln Z$  cannot be symmetric under  $\beta \rightarrow -\beta$ .

#### IV. PHASE TRANSITIONS

The partition function can be considerably simplified if all bonds have the same strength  $J_{ij} = \pm 1$ . In this case, a factor of  $\tanh \beta$  can be pulled out of the hopping matrix to yield  $W(\beta) = W \tanh \beta$ , where  $W$  depends only on the Ising model's connectivity and not on  $\beta$ . In this section, we develop an intuitive understanding for the mechanism of the phase transition by considering  $W$ 's spectrum in the complex plane. Although we have specialized to the case where all bond strengths are equal in magnitude, the basic ideas in this section are still general. With this

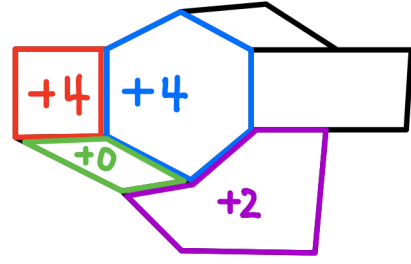


FIG. 5: If a planar Ising model can be drawn by joining polygons with an even number of sides, then all closed diagrams have an even number of bonds. This can be proved by recursively constructing closed diagrams through the merger of plaquettes. For example, here we have constructed a closed diagram by merging the red, blue, green, and purple plaquettes in that order. As a result, the partition function is symmetric in  $\beta$ , and the spectrum of its hopping matrix is invariant under  $\lambda \rightarrow -\lambda$ .

simplification, the partition function can be re-written as:

$$\ln Z = \ln Z_0 + \frac{1}{2} \sum_{\lambda} \ln(1 - \lambda \tanh \beta), \quad (13)$$

where  $\lambda$  is now a  $\beta$ -independent eigenvalue of  $W$ . Note that if we treat Eq. 13 simply as the definition of a function  $Z$ , then the values of  $\beta$  such that  $\beta = \text{artanh}(1/\lambda)$  are the zeros of  $Z$ . Of course, the partition function for real  $\beta$  is always positive, but its analytic continuation to complex  $\beta$  can be zero. As  $\beta$  ranges from 0 to  $\infty$ ,  $\tanh \beta$  ranges from 0 to 1, so it is the eigenvalues near the interval of the real line  $(1, \infty)$  that strongly affect the behavior of  $\ln Z$ . Because the partition function of a finite system is analytic and positive for all real  $\beta$ , the eigenvalues cannot lie exactly on  $(1, \infty)$ , but they are allowed to lie close with a small imaginary part (eigenvalues can accumulate at exactly 1, because  $\tanh(\beta) = 1$  is only attained at infinite  $\beta$ ). As the size of the system approaches infinity, the eigenvalues accumulate closer and closer to the real line until they are infinitesimally close in the thermodynamic limit, and the partition function becomes nonanalytic as  $\tanh \beta$  passes through these eigenvalues. For example, a plot of the hopping eigenvalues for a finite  $40 \times 40$  square lattice Ising model are shown in Fig. 6.

It is also possible for an Ising model to be critical at zero temperature due to frustration. The signature of such behavior is the accumulation of eigenvalues at exactly 1 (or  $-1$  for the model with opposite couplings). Although we would need to calculate the spin-spin correlation functions and show that they have algebraic decay to rigorously justify criticality at zero temperature, our eigenvalue picture can still provide some insight. Keeping the above assumptions that all couplings have equal magnitude, we can calculate the internal energy  $U = \langle -\sum_{\langle i,j \rangle} J_{ij} \sigma_i \sigma_j \rangle$  by differentiating the free

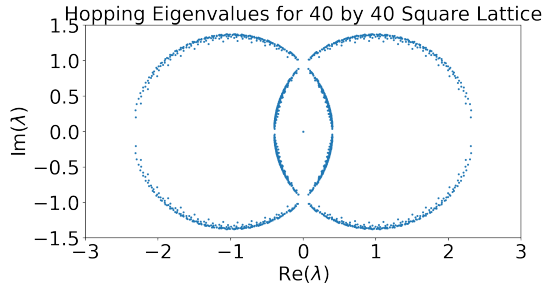


FIG. 6: The hopping eigenvalues for a  $40 \times 40$  ferromagnetic square lattice. There are no eigenvalues exactly on the real intervals  $(1, +\infty)$  and  $(-\infty, -1)$ , but as the system grows larger eigenvalues can accumulate *close* to those intervals and eventually touch the real line in the thermodynamic limit, providing a mechanism for nonanalytic behaviour. At this finite size, we already see eigenvalues accumulating near  $\sqrt{2} + 1 \approx 2.41$ , which corresponds to the exact critical temperature of  $\ln(\sqrt{2} + 1)/2$ .

energy:

$$-\frac{\partial \ln Z}{\partial \beta} = -N_b \tanh \beta + \frac{1}{2} \sum_{\lambda} \frac{\lambda(1 - \tanh^2 \beta)}{1 - \lambda \tanh \beta}. \quad (14)$$

In the low temperature limit,  $\tanh \beta \rightarrow 1$ , so the first term is simply  $N_b$ . The limit of the summand is a little subtle. If  $\lambda \neq 1$ , even by a tiny amount, the limit as  $\beta \rightarrow \infty$  is  $\lambda(1 - 1^2)/(1 - \lambda) = 0$ . However, if  $\lambda$  is exactly 1, then the limit must be evaluated using L'Hôpital's rule and turns out to be 2. Therefore, letting  $N(1)$  be the number of eigenvalues that are exactly 1, the internal energy at zero temperature is:

$$U = -N_b + N(1). \quad (15)$$

If there were no eigenvalues at 1, the internal energy would just be  $-N_b$ , indicating that all bonds are satisfied and the system is not frustrated. Because each frustrated bond incurs an energetic penalty of 2, the number of frustrated bonds is  $N(1)/2$ . To measure the ground state degeneracy, we can compute the limit of the entropy as  $\beta \rightarrow \infty$ . The entropy is given through the partition function as:

$$S = \ln Z - \beta \frac{\partial \ln Z}{\partial \beta} \quad (16)$$

Substituting our expression for  $\partial \ln Z / \partial \beta$  yields:

$$S = N_s \ln(2) + N_b (\ln \cosh \beta - \beta \tanh \beta) + \frac{1}{2} \sum_{\lambda} \left[ \ln(1 - \lambda \tanh \beta) + \frac{\beta \lambda (1 - \tanh^2 \beta)}{1 - \lambda \tanh \beta} \right]. \quad (17)$$

The limit as  $\beta \rightarrow \infty$  of  $(\ln \cosh \beta - \beta \tanh \beta)$  is  $-\ln 2$ . If  $\lambda \neq 1$ , the limit of the summand is just  $\ln(1 - \lambda)$ . However, if  $\lambda$  is exactly 1, the limit of the summand must

be evaluated using L'Hôpital's rule and is  $\ln 2$ . Therefore, the entropy at zero temperature is:

$$S = (N_s - N_b + \frac{N(1)}{2}) \ln 2 + \frac{1}{2} \sum_{\lambda \neq 1} [\ln(1 - \lambda)]. \quad (18)$$

Unfortunately, we cannot say anything more than this in general because the eigenvalues  $\lambda$  are unknown. The problem is that a nonzero residual entropy is not necessarily an indication of frustration. For example, a fully ferromagnetic Ising model could have  $C$  mutually-disconnected fragments. One can flip the spin of each component without changing the total energy, leading to a nonzero residual entropy of  $C \ln 2$ , despite the lack of frustration and hence zero  $N(1)$ .

## V. DUALITY

The dual  $\bar{G}$  of a planar graph  $G$  is constructed by converting all plaquettes of the original graph into vertices and connecting two vertices if their corresponding plaquettes shared an edge. For example, the square lattice is dual to itself, and the triangular and hexagonal lattices are dual to each other as shown in the Fig. 7. Closed dia-

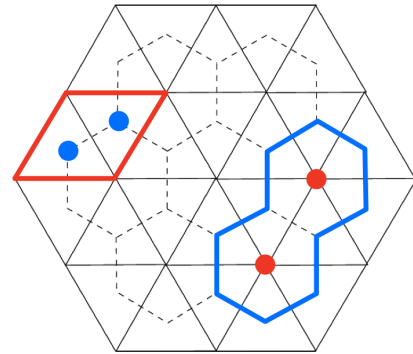


FIG. 7: The triangular lattice (solid line) and hexagonal lattice (dashed line) are dual to each other. A closed diagram on the triangular lattice (red line) is the boundary of an island on the hexagonal lattice (blue dots), and a closed diagram on the hexagonal lattice (blue line) is the boundary of an island on the triangular lattice (red dots).

grams on  $G$  form the boundaries of islands of spins on  $\bar{G}$ , and the boundaries of islands of spins on  $G$  form closed diagrams on  $\bar{G}$ . By thinking of these islands of spins as being flipped relative to a ground state background,  $G$ 's high-temperature expansion can be mapped to  $\bar{G}$ 's low-temperature expansion, which can be constructed by perturbing around the ground state (for a ferromagnetic model, this would be the fully spin-polarized state). The energy cost of an island of flipped spin in  $\bar{G}$  is:

$$\Delta H = \sum_{\langle \bar{i}\bar{j} \rangle \in \text{boundary}} 2|J_{\bar{i}\bar{j}}|, \quad (19)$$



where dual bond  $\bar{i}\bar{j}$  being on the boundary means that it connects a flipped spin in the island with a non-flipped spin in the background. Every bond  $ij$  in  $G$  is associated with the unique bond  $\bar{i}\bar{j}$  in the dual  $\bar{G}$  that crosses it, so the bond strengths of the dual model are given by:

$$\exp(-2\beta J_{\bar{i}\bar{j}}) = \tanh(\beta J_{ij}). \quad (20)$$

One may already see a problem with this duality. The LHS of Eq. 20 is positive, but the RHS can be negative, so there may be difficulties in constructing the dual of an Ising model with antiferromagnetic bonds. More generally, the low-temperature expansion assumes that there is a unique ground state to perturb around, and all terms in the expansion go as  $\exp(-\beta\Delta H_{\text{island}}) > 0$ . At low temperatures, frustrated models lack a unique ground state to perturb around, making it impossible to match terms with a high temperature expansion which always starts with  $1 + O(t)$ . At high temperatures, the expansion in powers of  $t$  for models with antiferromagnetic couplings can have terms of negative sign, which are never found in a low-temperature expansion in islands of flipped spin. Thus, the Kramers-Wannier duality of the square lattice Ising model can be generalized to general planar Ising models, but only if all their bonds are ferromagnetic.

## VI. SOME PERIODIC MODELS

We conclude our discussion by applying the formalism that we've developed to two important periodic lattices. For all periodic models, the directed edges can be indexed as  $|x, y, \mu\rangle$ , where  $x$  and  $y$  give the edge's unit cell and  $\mu$  denotes the particular type of edge inside the cell. For a model with  $N_x$  by  $N_y$  unit cells, we can Fourier transform over the  $x$  and  $y$  indices to block-diagonalize the hopping matrix:

$$\ln Z = \ln Z_0 + \frac{1}{2} \sum_{\mathbf{q}} \ln \det(1 - W(\mathbf{q}) \tanh \beta), \quad (21)$$

where the summation runs over all  $\mathbf{q} = (\frac{2\pi n_x}{N_x}, \frac{2\pi n_y}{N_y})$  for  $n_{1,2} = 0, \dots, N_{1,2} - 1$ . The matrix  $W(q)$  is defined in the same way as in Eq. 8, but there is an additional factor of  $\exp(-i\Delta n_x q_x - i\Delta n_y q_y)$  when the hopping changes the unit cell by  $\Delta n_x$  and  $\Delta n_y$ . The procedure here is very similar to diagonalizing tight-binding Hamiltonians in condensed matter theory. Imposing periodic boundary conditions as we have done is technically wrong because it breaks planarity, but this issue washes out when we take thermodynamic limit and replace sum with integral.

For the triangular lattice, with the assignment of edges

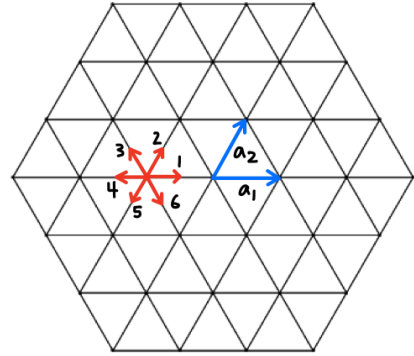


FIG. 8: Our choice of lattice vectors and labelling scheme for the directed edges in the triangular lattice Ising model.

and lattice vectors given by Fig. 8,  $W(\mathbf{q})$  is:

$$W(\mathbf{q}) = \begin{bmatrix} 1 & e^{-\frac{i\pi}{6}} & e^{-\frac{i\pi}{3}} & 0 & e^{+\frac{i\pi}{3}} & e^{+\frac{i\pi}{6}} \\ e^{+\frac{i\pi}{6}} & 1 & e^{-\frac{i\pi}{6}} & e^{-\frac{i\pi}{3}} & 0 & e^{+\frac{i\pi}{3}} \\ e^{+\frac{i\pi}{3}} & e^{+\frac{i\pi}{6}} & 1 & e^{-\frac{i\pi}{6}} & e^{-\frac{i\pi}{3}} & 0 \\ 0 & e^{+\frac{i\pi}{3}} & e^{+\frac{i\pi}{6}} & 1 & e^{-\frac{i\pi}{6}} & e^{-\frac{i\pi}{3}} \\ e^{-\frac{i\pi}{3}} & 0 & e^{+\frac{i\pi}{3}} & e^{+\frac{i\pi}{6}} & 1 & e^{-\frac{i\pi}{6}} \\ e^{-\frac{i\pi}{6}} & e^{-\frac{i\pi}{3}} & 0 & e^{+\frac{i\pi}{3}} & e^{+\frac{i\pi}{6}} & 1 \end{bmatrix} D(\mathbf{q}),$$

$$D(\mathbf{q}) = \text{diag}(e^{-iq_x}, e^{-iq_y}, e^{i(q_x - q_y)}, e^{iq_x}, e^{iq_y}, e^{i(-q_x + q_y)}), \quad (22)$$

where we have split  $W(\mathbf{q})$  into a part contributed by the winding of the tangent vector and a part coming from the Fourier transform.  $W(\mathbf{q})$ 's spectrum can be found analytically through a computer algebra system (CAS). For any value of  $\mathbf{q}$ , two eigenvalues are always at  $-1$ , and the other 4 are elsewhere in the complex plane. The other four eigenvalues depend on  $\mathbf{q}$  in a very complicated way, so we have omitted their explicit expressions. Nevertheless, the fact that there are  $2N$  eigenvalues at  $-1$  for a model with  $N$  unit cells allows us to conclude that one out of every three bonds in the antiferromagnetic ground state is frustrated. The triangular lattice has one spin per and three bonds per unit cell, so for a model of  $N$  unit cells the trivial part of the free energy is merely  $\ln Z_0/N = \ln 2 + 3 \ln \cosh \beta$ . Computing  $\ln \det(1 - W(\mathbf{q}) \tanh \beta)$  is best left to a CAS, and the density of wavevectors for a model with  $N$  unit cells is  $N/(2\pi)^2$ , so the final partition function for the triangular lattice is given by Eq. 24 (the equation spans the whole page, so we have placed it at the end to avoid disrupting the formatting), where  $t = \tanh \beta$  for brevity. This integral over wavevectors has singular behavior when the quantity inside the logarithm is zero, which occurs in two cases. Firstly, for all values of  $\mathbf{q}$ , the quantity inside the logarithm is 0 as  $\tanh \beta \rightarrow -\infty$ , indicating yet again that the antiferromagnetic Ising model is critical at zero temperature. Secondly, although not obvious from its messy expression, the argument of the logarithm is also zero at  $t = 2 - \sqrt{3}$  and  $\mathbf{q} = \mathbf{0}$ , indicating that the ferromagnetic Ising model's critical temperature is  $1/\text{artanh}(2 - \sqrt{3}) = 4/\ln 3$ .

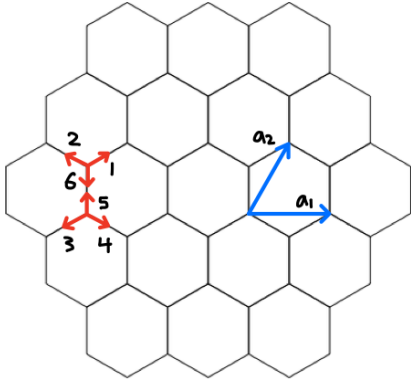


FIG. 9: Our choice of lattice vectors and labelling scheme for the directed edges in the hexagonal lattice Ising model.

For the hexagonal lattice, with the assignment of edges and lattice vectors given by Fig. 9,  $W(\mathbf{q})$  is:

$$W(\mathbf{q}) = \begin{bmatrix} 0 & 0 & 0 & e^{+\frac{i\pi}{6}} & e^{-\frac{i\pi}{6}} & 0 \\ 0 & 0 & e^{-\frac{i\pi}{6}} & 0 & e^{+\frac{i\pi}{6}} & 0 \\ 0 & e^{+\frac{i\pi}{6}} & 0 & 0 & 0 & e^{-\frac{i\pi}{6}} \\ e^{-\frac{i\pi}{6}} & 0 & 0 & 0 & 0 & e^{+\frac{i\pi}{6}} \\ e^{+\frac{i\pi}{6}} & e^{-\frac{i\pi}{6}} & 0 & 0 & 0 & 0 \\ 0 & 0 & e^{+\frac{i\pi}{6}} & e^{-\frac{i\pi}{6}} & 0 & 0 \end{bmatrix} D(\mathbf{q}),$$

$$D(\mathbf{q}) = \text{diag}(e^{-iq_y}, e^{i(q_x - q_y)}, e^{iq_y}, e^{i(-q_x + q_y)}, 1, 1), \quad (23)$$

where we have again split  $W(\mathbf{q})$  into a part contributed by the winding of the tangent vector and a part coming from the hopping between different cells. Similarly to the triangular lattice,  $W(\mathbf{q})$ 's spectrum can be found analytically through a CAS, although again the expressions are complicated enough to make them not worth including explicitly. The hexagonal lattice has two spins and three bonds per unit cell, so for a model of  $N$  unit cells the partition function is Eq. 25 (the equa-

tion spans the whole page, so we have placed it at the end to avoid disrupting the formatting), where as before  $t = \pm \tanh \beta$ . The argument of the logarithm becomes zero when  $\mathbf{q} = \mathbf{0}$  and  $t = 1/\sqrt{3}$ , indicating that the hexagonal lattice Ising model's critical temperature is  $1/\text{artanh}(1/\sqrt{3}) = 2/\ln(2 + \sqrt{3})$  (the ferromagnetic model and antiferromagnetic models are equivalent because the hexagon has six sides). In both the triangular and hexagonal lattices, the partition function has a similar behavior as we approach the critical point, demonstrating the universality of the transition.

## VII. CONCLUSION

In this paper, we derived the exact combinatorial solution to general planar Ising models. Using the exact solution, we demonstrated that bulk thermodynamic quantities of planar models have polynomial complexity, examined the mechanism for phase transitions in the thermodynamic limit, and explored frustration and duality. Finally, we closed off our analysis by computing the partition functions and critical temperatures of the triangular and hexagonal lattice Ising models.

## Acknowledgements

The author is very grateful to Professor Mehran Kardar for introducing the wonders of statistical mechanics such as phase transitions and universality in an incredible semester, as well as for helpful discussions regarding this paper. Aside from the author's derivation of the combinatorial solution being based on M. Kardar's presentation of the square lattice's exact solution [1], the author did all derivations in this work originally as an intellectual exercise.

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[1] M. Kardar, *Statistical Physics of Fields* (Cambridge University Press)

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## Supplementary Equations

The partition functions for the triangular and hexagonal lattices are given below.

$$\frac{\ln Z}{N} = \ln(2 \cosh^3 \beta) + \frac{1}{2} \int \frac{d^2 \mathbf{q}}{(2\pi)^2} \ln \left[ (t+1)^2 (1 - 2t + 6t^2 - 2t^3 + t^4 - 2t(-1+t)^2 (\cos(q_x) + \cos(q_x - q_y) + \cos(q_y))) \right] \quad (24)$$

$$\frac{\ln Z}{N} = \ln(2^2 \cosh^3 \beta) + \frac{1}{2} \int \frac{d^2 \mathbf{q}}{(2\pi)^2} \ln \left[ 1 + 3t^4 + 2t^2(-1+t^2) (\cos x + \cos(x-y) + \cos(y)) \right] \quad (25)$$