Topological defects in an MHD fluid

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May 19, 2023

Abstract

The divergence-free nature of magnetic fields implies that fluctuations of the field must take the form of closed loops. By considering these loops as topological defects in an otherwise unmagnetized MHD fluid, a partition function can be constructed for an ensemble of flux loops. The degree of linkage increases with temperature and above a certain critical temperature a phase change occurs as the average helicity per particle of the ensemble diverges until particle size saturates at the system size. The system becomes helically magnetized as a result of thermal fluctuations.

1 Introduction

Many astronomical objects are composed of highly magnetized plasma. Planets, stars, black holes, neutron stars, even the interstellar medium all have large scale magnetic structure. The magnetic fields of these objects appear to exhibit large scale order and small-scale turbulent fluctuations. The mechanism through which these large-scale fields are generated and sustained is not well understood [1]. Recent efforts have shown that kinetic scale instabilities in a shear-flow plasma can generate small scale filamentary magnetic fields that lead to spontaneous magnetization, but have yet to explain how this magnetization is amplified to the strength observed or how it is sustained [2]. This work explores the possibility of a statistical approach to the question of cosmic magnetogenesis and spontaneous magnetization.

Through successive averaging and coarse graining of the Vlassov equation, which describes the evolution of kinetic populations of ions and electrons, one can derive the set of magnetohydrodynamic (MHD) fluid equations [3]. These equations describe a conductive fluid with fluid density and bulk flows defined by the ion species and a current due to the motion of the electron population relative to the ions. The MHD equations have been very well studied and are useful for describing of a variety of systems, from fusion plasmas to black hole accretion disks to the molten core of the Earth [4]. In the so-called Ideal MHD limit, the resistivity is neglected and the time evolution of the magnetic field is given by

$$\frac{\partial B}{\partial t} = \nabla \times (v \times B) \tag{1}$$

from which it can be shown that the magnetic field lines are essentially ‘frozen in’ to the fluid and can be understood to move with the fluid as it flows. Because v is a continuous field, the topology of field lines is invariant under fluid motions. If the resistivity cannot be neglected, then this is no longer true and field lines can break and reconnect, altering field line topology [5].

Because the magnetic field is divergence free, field lines must form closed loops. This constraint motivates the study of magnetic fields in terms of the topology of field line loops (ie the linkage, twist, and writhe of loops) [6]. It is useful to conceptualize the magnetic field as being composed of flux ropes that contain a definite amount of magnetic flux within a tubular boundary. Flux ropes are well behaved objects in MHD and make topological consideration of the field much clearer. An interesting and useful quantity in the study of flux ropes is the magnetic helicity:

$$H = \int d^3\vec{x}\vec{A} \cdot \vec{B} \tag{2}$$

where $\vec{A}$ is the magnetic vector potential and $\vec{B}$ is the magnetic field. Helicity is a measure of the topological structure of flux ropes and can be decomposed into internal helicity due to twist and writhe, and external helicity due to linkages amongst flux ropes [6]. In Ideal MHD, the helicity of a system is a conserved quantity as the system evolves, and furthermore, the internal and external helicities are separately conserved because of the topological invariance of flux ropes. Allowing a finite resistivity, internal and external helicity can be exchanged via reconnection, but the total magnetic helicity remains approximately conserved. In fact, the total magnetic helicity is better conserved than the magnetic energy when resistive dissipation is considered [7].

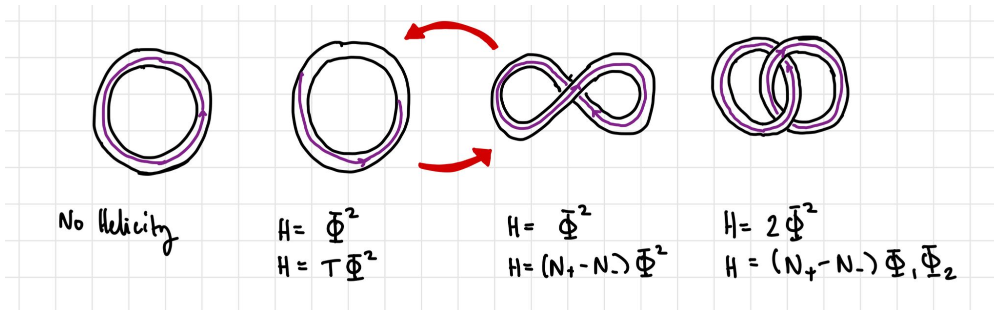

Figure 1: These diagrams represent the simplest helical configurations of loops. A characteristic field line is drawn in purple on the the toroidal surface of a flux rope loop. Net helicity is generated based on the twisting of flux ropes, T, which is the ratio of how many times a field line goes around toroidally vs poloidally, and the crossing number defined to be positive or negative based on a right hand rule. Internal helicity can change form from twist to crossing helicity in topologically allowed deformations.

The invariance of total helicity imposes a constraint on how the magnetic field evolves. The Woltjer-Taylor Relaxation Theorem says that an MHD system will relax to the minimum energy state for a given amount of magnetic helicity, and in this state helical structures relax to the largest spatial scale available. The minimum energy state is characterized by being 'force-free', satisfying $\nabla \times \vec{B} = \lambda \vec{B}$ [8][5]. This relaxation has been observed in laboratory plasmas, and in astrophysical contexts [9]. Numerical studies of flux rope dynamics show that an initially random magnetic field will evolve via an 'inverse cascade' where magnetic energy is transferred to larger spatial scales, saturating at the system scale [10]. This work explores a statistical field theoretic formulation of this phenomena.

2 Magnetic fluctuations in a conductive fluid

In this calculation, the system under consideration is an MHD fluid of outer dimension L. The fields present in the fluid are the mass density, ρ, the velocity, $\vec{v}$, the thermal pressure, P, the electric current, $\vec{j}$, and the electric and magnetic fields, $\vec{E}$ and $\vec{B}$. The statistical weight of a particular configuration of these MHD fields can be written as a Boltzmann Weight by considering the various forms of energy manifest in these fields. The partition function can then be calculated by performing a functional integral of this weight over all possible configurations of the fields:

$$Z = \int \mathcal{D}\rho \mathcal{D}\vec{v} \mathcal{D}P \mathcal{D}\vec{j} \mathcal{D}\vec{E} \mathcal{D}\vec{B} e^{-\beta \mathcal{H}[\rho, \vec{v}, P, \vec{j}, \vec{E}, \vec{B}]} \tag{3}$$

Where $\mathcal{H}$ represents the energy functional on the fields and $\beta = (kBT)^-1$ is the usual temperature factor. As we are interested in the behavior of the magnetic field, we will make a simplifying assumption that the energy of a configuration of fields can be factored so that $\mathcal{H}[\rho, \vec{v}, P, \vec{j}, \vec{E}, \vec{B}] = \mathcal{H}_0[\rho, \vec{v}, P, \vec{j}, \vec{E}] + \mathcal{H}_B[\vec{B}]$, allowing a separation of the partition function as $Z = Z_0 Z_B$ and isolated study of the statistical properties of the magnetic field. This is valid if the system has relaxed to a force-free state. In this case the magnetic energy can be written as:

$$-\beta \mathcal{H}_B = \int d^3\vec{x} \frac{t}{2} B^2 + \frac{k}{2} \vec{B} \cdot \nabla \vec{B} \tag{4}$$

The first term represents the magnetic energy density of the field, and the second represents the energy associated with the magnetic tension of field lines that acts to straighten field lines. The coefficients t and k are phenomenological in nature and allow us to treat parametric dependencies on things like temperature.

2.1 Loop topological defects

An initially uniform, unmagnetized system will develop magnetic fluctuations as temperature increases. These fluctuations can develop via well known fluid and/or kinetic mechanisms such as the Biermann or Weibel effects. Fluctuation fields must form closed loops, and so a phenomenological theory can be developed by treating these fluctuations as loop topological defects. In an MHD fluid, monopolar defects are unphysical, so loops are the only defects that appear. A defect can be defined by the magnetic flux, Φ, contained within a flux rope which closes on itself. The magnetic tension of the defect is minimized when the loop is circular and untwisted, and the magnetic pressure associated with flux contained in the loop is minimized for a circular cross section. Therefore, the lowest energy topological defect can be considered as a circular torus with major radius R and minor radius a.

The energy of such a defect can be calculated directly by integration over the toroidal domain of the defect, yielding $\beta E_0 = \frac{\Phi^2}{\mu_0 a^2}(Rt + k(a))$. The tension constant is taken to be a function of the minor radius on phenomenological grounds, as a thicker tube is harder to bend into a circle. Note that the cost of forming a defect is finite so loops can appear at any temperature with the appropriate Boltzmann weight, $W_1 = \frac{L^3}{2\pi^2 Ra^2} e^{-\beta E_0}$. The prefactor represents the configurational entropy due to the fact that the loop can form anywhere in the system. Flux loops form with a continuum of sizes and shapes, with deviation from a circle increasing with temperature.

A system with total average magnetic flux due to thermal fluctuations $\langle\Phi\rangle$ can be considered as a gas of N loop defects containing flux $\delta\Phi$ so that $\langle\Phi\rangle = \sum^N \delta\Phi$. For simplicity, we consider the gas to be composed of untwisted flux loops of uniform size that increases with temperature [1]. If the flux loops are distributed randomly in space, the probability that two will form as a link is $\mathcal{P}_{link} = \frac{2\pi^2 R^3}{L^3} e^{-\beta E_0[\delta\Phi]}$ as the volume that a second loop could be in and form a link is a torus of major and minor radius R (because the relative orientation of loops matters, this is an overestimation of the probability). Since all flux is contained within the flux ropes, there is no interaction energy between loops, and the energy cost is that of forming two individual loops. The helicity of a link of two loops is $\pm 2(\delta\Phi)^2$. A linkage will be considered as a single particle in the gas of size $2R$ because conservation of helicity requires they stay linked.

At low temperatures, the average radius is small compared to the system size so it is unlikely that links will be formed, little helicity is generated, and the system remains unmagnetized. The unlinked loops will move around the system and have a probability of colliding and generating helicity through reconnective interactions, but exploring this is beyond the scope of this study[2].

As temperature increases, the size of loops will increase and configurations that include sets of linked loops become more probable. The average number of links per particle can be estimated by considering the expected number of pairwise linkages of n loops:

$$\langle N_n \rangle = \frac{n-1}{n} \mathcal{P}_{link}^{n-1} \quad (5)$$

The factor of $n-1$ accounts for the fact that larger linkages are larger and so it is more likely that additional loops will form links, and the denominator is present because each loop of length n counts as one particle in the gas. Links could also form between more than two loops at a time, yielding more complex linkages[3].

[1] Higher order defects that include twisted and writhed loops should also appear with a higher energy cost than the simple defects considered here. These defects have internal helicity, and form toroidally knotted structures above 3 units of helicity. Considering these defects should lower the critical temperature because the formation of linked knots enables even faster helicity growth as the size of defects increases. Calculating the energy cost of these knotted defects is left to future investigation because of the complexity in determining the minimum energy configuration of a knotted flux rope.

[2] Including reconnection and higher order defects together will lower the critical temperature further as it opens up the topological class of satellite knots which are knots formed by linking together two or more knots. Reconnection should be most important in the low temperature limit as resistivity of a plasma scales $T^{-3/2}$ and so low temperature plasmas are further from the ideal limit.

[3] At the level of 3 loop linkages, the Borromean rings appear as a set of 3 loops that have no linkages, and yet are bound together. I don't know how to treat this case, but it is interesting to consider.

Considering only pairwise links, the average number of links per particle is $\langle n \rangle = \frac{1}{N} \sum_n \langle N_n \rangle n = \frac{1}{N} \sum (n-1)\mathcal{P}_{link}^{n-1}$, and the average helicity per particle is $\langle H \rangle = 2(\delta\Phi)^2 \langle n \rangle$.

Above a critical radius $R_c = \left(\frac{L^3 e^{2\beta E_0}}{2N\pi^2} \right)^{1/3}$, the statistical weight of linked particles increases rapidly, and so too does the average helicity per particle of the gas. The helical linkages will relax to the largest scale they can for fixed energy, meaning that the characteristic size of particles in the gas also increases, further increasing the probability of forming more highly linked collections of loops. As more loops become linked, more and more helical configurations become accessible. The average helicity per particle grows exponentially fast until particle size saturates at the system size. The gas becomes more connected as well, and there is an analogy to problems involving percolation. At this point, the system is fully magnetized with helical structure that could in principle be calculated by detailed consideration of the statistical weights of all possible linkages.

3 Discussion

A renormalization group could be constructed for the above ensemble of loop defects near the critical temperature. One could coarse grain the system by defining a new set of flux rings scaled by a factor b and sum up the total helicity within these new boundaries. Distances could then be rescaled so that the radius of loops is kept constant, and the flux in each loop could then be renormalized so that the total magnetic flux remains constant. Methods have been established for computing the helicity of a system where flux ropes cross the boundary of the region of integration [Berger '83]. The helicity within the coarse-grained loops could be treated as internal helicity while the helicity crossing boundaries contributes to external helicity. This coarse-grained ensemble could then be analyzed to understand the interlinkages between different parts of the system. Treating loops with internal helicity requires the inclusion of knots in the analysis and reconnective interactions, both of which are beyond the scope of this work.

The above description of a gas of loops of increasing helicity with temperature is a low temperature expansion of the magnetic topology. One could also consider a high temperature expansion. In this limit, a flux rope could be considered as taking a random walk through the system. In 3 dimensions, a random walker will never return to its starting position, but it will generate helicity as it twists and turns and passes through loops in its path. The helicity of individual random flux ropes is not well defined, but integrating over the entire region gives the total helicity of the system.

Because helicity is well conserved as structures relax to a minimum energy state, I thought about defining an effective energy in terms of the helicity, but ultimately, I couldn't determine a way to formulate this. I think this is an idea to pursue further in the future, as it may provide straight forward way to include arbitrary helical structures like knots, hyperbolic links, and satellite links that avoids explicit calculation of the energy associated with bending and twisting flux ropes. There is literature describing the challenges of calculating the lowest energy configurations of topological ropes, like in DNA supercoils [11] and physical rope knots [12], but there is as of yet no general theory. It would be convenient to use helicity itself as an effective energy for Boltzmann weights, but I could not think of how to justify this.

In conclusion, this investigation has shown that loop topological defects of the magnetic field can form in an MHD fluid as a result of thermal fluctuations because they have a finite energy of formation. These loops can be treated as an ensemble and consideration of the average helicity per particle as a function of temperature suggests the existence of a phase transition between a low temperature unmagnetized state and a high temperature helically magnetized state. Much work remains to rigorously demonstrate this, including accounting for internal helicity manifesting as knots and allowing reconnective interactions. Whether or not these loop topological defects exist as observable entities, they are an interesting conceptual basis with which to consider the structure of magnetic fluctuations and the evolution of helicity in MHD fluids.

References

[1] L. M. Widrow. Origin of galactic and extragalactic magnetic fields. *Rev of Mod Phys*, 74, 2002.

[2] M. Zhou. Spontaneoud magnetization of collisionless plasma. *PNAS*, 2022.

[3] D. R. Micholson. *Introduction to Plasma Theory.* John Wiley and Sons, 1983.

[4] J. Freidberg. *Ideal MHD.* Cambridge University Press, 2014.

[5] J. B. Taylor. Relaxation of toroidal plasma and generation of reverse magnetic fields. *PRL*, 1974.

[6] M. A. Berger. The topological properties of magnetic helicity. *J. Fluid Mech.*, 1983.

[7] E. G. Blackman. How spectral shapes of magnetic energy and magnetic helicity influence their respective decay timescales. *Plasma Phys Cont Fus*, 2004.

[8] L. Woltjer. A theorem on force free magnetic fields. *PNAS*, 1958.

[9] P. M. Bellan. *Spheromaks.* Imperial College Press, 2000.

[10] M. Zhou. Multi-scale dynamics of magnetic flux tubes and inverse magnetic energy transfer. *JPP*, 2020.

[11] F. B. Fuller. The writhing number of a space curve. *PNAS*, 1971.

[12] V. P. Patil. Topological mechanics of knots and tangles. *Science*, 2019.