Phase separation and detailed balance violation in a two-temperature system

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We examine time-reversal-symmetry breaking across scales in a two-temperature system, where two species of brownian particles interact while maintaining contact with different thermal reservoirs. Particles coupled in this way have been seen to phase separate, with cold particles forming dense clusters surrounded by a dilute gas of hot particles. While the equilibrium theory seems to describe the phenomenology of the phase separation, this phase separation in the absence of attractive interactions is an inherently out-of-equilibrium phenomenon. Looking to salvage the lost non-equilibrium character of our system, we find that detailed balance is indeed broken at both the microscopic scale and on phase interfaces at the macroscopic scale. We observe ratchet currents when two particles of different temperatures are placed in an asymmetric landscape, via a mechanism similar to ratchet currents in self-propelled particles.

I. INTRODUCTION

In the real world, broken thermal equilibrium is the rule rather than the exception. In recent years, there has been an explosion of research on the statistical mechanics of out-of-equilibrium systems. One very active area of research is that of active matter, where thermal equilibrium is violated at the microscopic scale by injecting individual degrees of freedom with energy that allows them to self-propel [1], rotate [2], process information [3], and more.

One thing that non-equilibrium systems have in common, and which is impossible in equilibrium systems, is the existence of nonzero steady-state fluxes in their configuration spaces [4]. In active matter, such fluxes occur in the microscopic configuration spaces of individual degrees of freedom, and whether such fluxes survive coarse-graining to generate macroscopic motion (e.g. ratchet currents [5], shear cycles [6], etc.) is an important question.

One more subtle type of thermal equilibrium violation occurs when you take equilibrium systems at two different temperatures, and couple them while maintaining their coupling to their own thermal reservoirs. For instance, one may consider species A and B of pairwise interacting Brownian particles, governed by Langevin equations of motion with differing noise strengths:

$$\dot{x}_A^i = -\frac{\partial}{\partial x_A^i}\sum_{j=1}^{N_A} u_{AA}(x_A^i - x_A^j) - \frac{\partial}{\partial x_A^i}\sum_{k=1}^{N_B} u_{AB}(x_A^i - x_B^k) + \sqrt{2T_A}\eta_A^i \tag{1a}$$

$$\dot{x}_B^i = -\frac{\partial}{\partial x_B^i}\sum_{j=1}^{N_B} u_{BB}(x_B^i - x_B^j) - \frac{\partial}{\partial x_B^i}\sum_{k=1}^{N_A} u_{AB}(x_B^i - x_A^k) + \sqrt{2T_B}\eta_B^i. \tag{1b}$$

(Throughout this paper, we assume all particles have equal mobilities, which we set to 1.)

Such a system was investigated analytically by Refs. [7, 8] and through simulations by Refs. [9, 10]. It was found that the two species tended to phase separate, with hot particles forming a dilute gas surrounding clusters of cold particles. See a snapshot of our simulations near the critical point in Fig. 1. Remarkably, however, this phase separation can be described by the minimization of an effective equilibrium free energy.

It is known that many active matter systems possess effective equilibrium free-energy functionals that decently describe the dynamics at the macroscopic scale (e.g. Ref. [11]). However, there often remain signatures of broken thermal equilibrium such as non-vanishing entropy production rates near phase interfaces [12, 13] or ratchet currents in the presence of an external potential [5].

In this paper, we ask the question: what happened to the broken thermal equilibrium in the two-temperature system, and where can it be recovered? In Sec. II, we examine the effective equilibrium phase separation and compare existing analytic results [7, 8] with simulations [9]. In Sec. III we zoom into the micro-scale and investigate steady-state currents that occur in two-particle two-temperature systems, where we find the emergence of a ratchet current in the presence of an asymmetric external potential. An analogy is drawn between self-propelled particles and a pair of hot and cold particles. We then investigate the mesoscopic non-equilibrium terms that arise at the scale of the field dynamics once one considers 3-body interactions [8], and the consequential generation of mesoscopic currents.

II. PHASE SEPARATION: EFFECTIVE EQUILIBRIUM

In Refs. [7, 8], they consider a mixture of two species of particles, with two different temperatures. They integrate the master equation over all but one coordinate and obtain a hierarchy of equations, which they truncate by, crucially, neglecting three-body interactions. Then, they

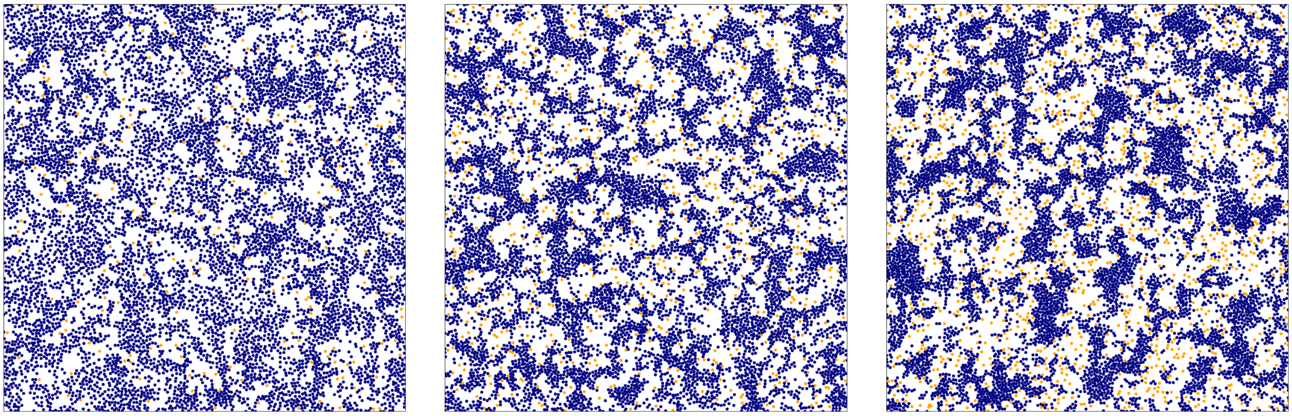

FIG. 1. Snapshots of particle configurations in simulations we performed of a binary mixture of hot (orange) and cold (dark blue) particles with a temperature ratio of 0.01, after a simulation time $t = 10^4 a^2/T_{\mathrm{H}}$. As the baseline concentrations N_H/L^2 and N_C/L^2 are increased, the particles pass from the disordered phase (left) through the critical point (middle) into the ordered phase (right). See Fig. 4 for a plot of the mean-field free energy at these points (yellow, dashed black, and purple respectively).

perform a gradient expansion on the density fields (converting a convolution with the force kernel to multiplication by force moments or "virial coefficients"). These calculations give a deterministic Model B (or Cahn-Hilliard) equation of motion for each species A interacting with members of all species, indexed by γ

$$\frac{\partial c_A(\vec{r},t)}{\partial t} = \nabla\left[c_A(\vec{r},t)\nabla\mu_A(\vec{r},t)\right] \tag{2a}$$

$$\mu_A = T_A \ln c_A + \sum_\gamma T_{A\gamma}\left[B_{A\gamma}c_\gamma + \Lambda_{A\gamma}\nabla^2 c_\gamma\right] \tag{2b}$$

with pairwise effective temperature

$$T_{A\gamma} = \frac{T_A + T_\gamma}{2} \tag{3}$$

and coefficients

$$B_{A\gamma} = \int (1 - e^{-u_{A\gamma}(\vec{r})/T_{A\gamma}})d\vec{r} \tag{4a}$$

$$\Lambda_{A\gamma} = \frac{1}{6}\int r^2(1 - e^{-u_{A\gamma}(\vec{r})/T_{A\gamma}})d\vec{r}. \tag{4b}$$

Such theories describe separation between two phases, e.g. between a polymer and a solvent. The fact that this phase separation happens, in the absence of attractive interactions, is a signature of the non-equilibrium nature of our system, much like motility-induced phase separation in self-propelled particle suspensions [11]. Indeed, we will see that the mechanism for such phase separation is due to the existence of nonzero microscopic fluxes in the joint configuration space of a pair of hot and cold particle, in the same way that there is a nonzero flux in the joint configuration space of a self-propelled particle's position and orientation.

Remarkably, these chemical potentials μ_A, μ_B can be expressed as the functional derivatives of an effective free energy $\mu_\gamma = \delta\mathcal{F}/\delta c_\gamma$, where

$$\mathcal{F} = \int d\vec{r}\sum_A \Big[T_A c_A(\ln c_A - 1) + \frac{1}{2}\sum_\gamma T_{A\gamma}\left(B_{A\gamma}c_A c_\gamma - \Lambda_{A\gamma}(\nabla c_A)(\nabla c_\gamma)\right)\Big]. \tag{5}$$

Consequently, at this level, there remains no trace of detailed balance violation. As a corollary, the system admits an equation of state (derived in Refs. [7, 8]), has zero entropy production (see Ref. [12]), and exibits no steady-state currents in configurational space. In particular, this theory predicts that this system will exhibit no macroscopic spatial currents or "ratchet currents," even in the presence of an asymmetric potential (which induces ratchet currents in self-propelled particles, e.g. Ref. [5]).

One can connect this with free energies obtained from top-down approaches by expanding around reference concentrations c_A^0 and c_B^0 and defining deviations $\phi_A = c_A - c_A^0$ and $\phi_B = c_B - c_B^0$. Keeping only up to 4th order in the ϕ's, we get a ϕ^4 field theory with two order parameters coupled at the quadratic order:

$$\mathcal{F} = \int d\vec{r}\sum_\gamma \Big[\frac{1}{2}t_\gamma\phi_\gamma^2 + \frac{1}{3!}w_\gamma\phi_\gamma^3 + \frac{1}{4!}u_\gamma\phi_\gamma^4 + \frac{1}{2}\sum_\delta\left(t_{A\delta}\phi_\gamma\phi_\delta + K_{\gamma\delta}(\nabla\phi_\gamma)(\nabla\phi_\delta)\right)\Big] \tag{6}$$

with parameters

$$t_A = T_A \left(B_{AA} + \frac{1}{c_A^0} \right), \quad t_{A\gamma} = T_{A\gamma} B_{A\gamma} \tag{7a}$$

$$K_A = -T_A \Lambda_{AA}, \quad K_{A\gamma} = -T_{A\gamma} \Lambda_{A\gamma} \tag{7b}$$

$$w_A = -\frac{T_A}{(c_A^0)^2}, \quad u_A = 2\frac{T_A}{(c_A^0)^3}. \tag{7c}$$

We now investigate the phase diagram of this theory.

A. Mean field phase diagram

First consider homogenous phases ($\nabla\phi_A = \nabla\phi_B = 0$) and ignore fluctuations. The condition of stability of a homogenous phase is

$$\frac{\partial f}{\partial \phi_A} = \frac{\partial f}{\partial \phi_B} = 0 \tag{8}$$

$$\frac{\partial^2 f}{\partial \phi_A^2} \frac{\partial^2 f}{\partial \phi_B^2} - \left(\frac{\partial^2 f}{\partial \phi_A \phi_B} \right)^2 > 0. \tag{9}$$

The first condition is always true in the disordered phase ($\phi_A = \phi_B = 0$). The spinodal line bounds the region where the free energy of the disordered phase is concave up. It is given by

$$0 < \left[\frac{\partial^2 f}{\partial \phi_A^2} \frac{\partial^2 f}{\partial \phi_B^2} - \left(\frac{\partial^2 f}{\partial \phi_A \phi_B} \right)^2 \right]_{\phi_A = \phi_B = 0} \tag{10}$$

$$\implies t_{AB}^2 < t_A t_B, \text{ or} \tag{11}$$

$$T_{AB}^2 B_{AB}^2 < T_A T_B \left(B_{AA} + \frac{1}{c_A^0} \right) \left(B_{BB} + \frac{1}{c_B^0} \right). \tag{12}$$

When this quantity is positive, the free energy is both stationary and stable at $\phi_A = \phi_B = 0$.

In Ref. [9], they have performed a suite of simulations with 300 hot and 300 cold particles, varying both the diffusivity and the system size (which causes variation in the average densities $c_A^0 = c_B^0$). Cold particles were observed to aggregate into typically one large cluster when they were cold enough and the density was high enough. The steady-state fractional occupation of the largest cluster was used as a proxy for the condensation order parameter $\phi_A - \phi_B$, and is plotted in their Fig. 2, displaying what appears to be a 1st-order phase transition with finite-size effects. We plot a "phase diagram" showing this order parameter along with the spinodal (eq. 12), above which condensation is expected. The agreement between analytic theory and simulations is satisfactory at best.

The critical point is the point (on the spinodal) where the spinodal is perpendicular to the zero eigenvector of the Hessian of f, which implies maximum fluctuations. In Ref. [8] they find this to be the point where

$$\frac{B_{AB} c_B^0 (1 + c_B^0 B_B)}{(1 + c_A^0 B_A)^2} = \frac{T_A}{T_{AB}} \tag{13}$$

$$\left(B_A + \frac{1}{c_A^0} \right) \left(B_B + \frac{1}{c_B^0} \right) = \frac{T_{AB}^2 B_{AB}^2}{T_A T_B}. \tag{14}$$

See Fig. 3 for Fig. 2a of Ref. [8], which depicts the triangular phase diagram of the hot particle, cold particle, and solvent mixture.

The free energy is plotted for baseline concentrations c_A^0, c_B^0 near the critical point in Fig. 4. Upon running simulations of many hot and cold particles while increasing baseline concentrations through the critical point, we find steady-state configurations departing from a homogenous state, through a scale-invariant regime, to a phase-separated state with clusters of a fixed size (Fig.1). This provides definitive, albeit qualitative, support for the analytic predictions of Refs. [7, 8].

B. Fluctuations in the Gaussian model

We would like to predict statistical quantities such as correlations between Fourier components of the density fields and possible fluctuation-induced changes to the phase diagram. We convert the theory to the fluctuating version by adding a noise term, $\sqrt{T_\gamma c_\gamma}\eta_\gamma(x,t)$, that arises naturally in the coarse-graining of the fluctuating fields (see Ref. [14]), which we omit for brevity.

We will make the simplifying assumption that the mobility, found to be c_A by our explicit coarse-graining, can be treated as constant $c_A \approx c_A^0$. This makes the noise additive, rather than multiplicative, which is very helpful. The equations of motion are then

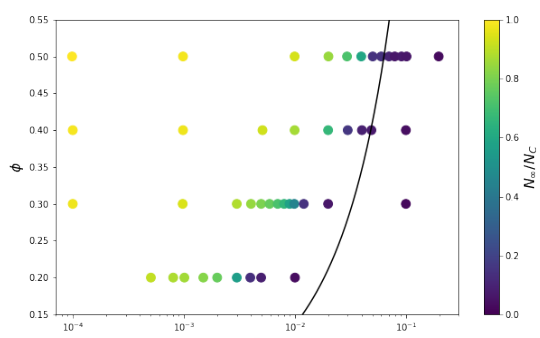

FIG. 2. Steady-state fractional occupation of the largest cluster in simulations of 300 hot and 300 cold particles from Ref. [9], varying the diffusivity ratio D and the total packing fraction $\phi = (c_A^0 + c_\beta^0)\pi a^2$. The spinodal (eq. 12) is plotted in black.

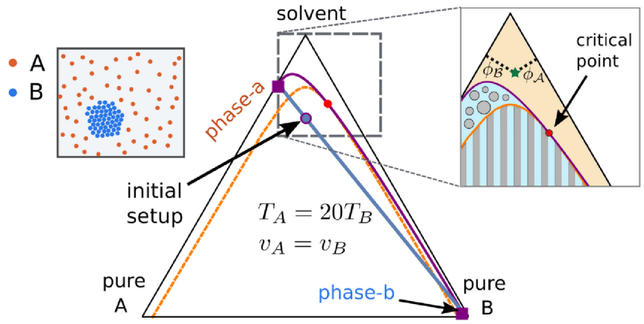

FIG. 3. Fig. 2a of Ref. [8] depicting the phase diagram of the two-temperature model

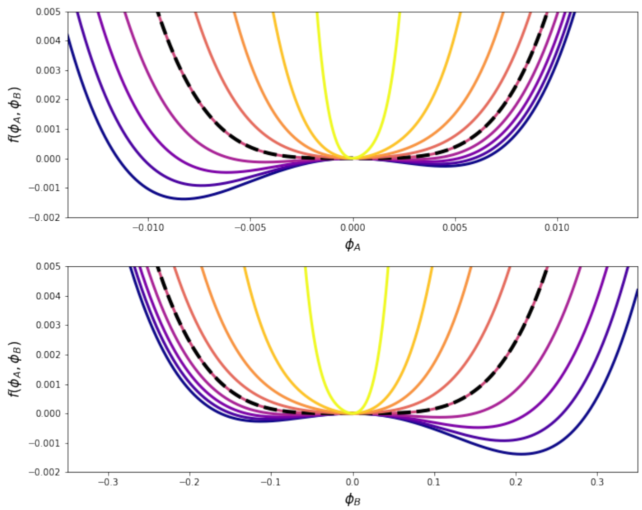

FIG. 4. Mean-field free energy along a slice of ϕ_A, ϕ_B phase space tangent to the spinodal (eq. 12), as the baseline concentrations c_A^0, c_B^0 are varied from the disordered phase (yellow) through the critical point (black dashed line) to the ordered phase (purple), on a phase space trajectory perpendicular to the spinodal.

$$\frac{\partial \phi_A}{\partial t} = \nabla \cdot \Bigg[c_A^0 \nabla \Big(\sum_\gamma \left(t_{A\gamma}\phi_\gamma - K_{A\gamma}\nabla^2 \phi_\gamma \right) + \frac{w_A}{2}\phi_A^2 + \frac{u_A}{3!}\phi_A^3 \Big) + \sqrt{T_A c_A^0}\,\eta_A(x,t) \Bigg], \quad (15)$$

where η_A is a standard Gaussian white noise with real- and Fourier-space correlation

$$\langle \eta_\gamma^i(x,t)\eta_\delta^j(x',t')\rangle = \delta^{ij}\delta_{\gamma\delta}\delta^d(x-x')\delta(t-t') \quad (16)$$
$$\langle \tilde{\eta}_\gamma^i(q,\omega)\tilde{\eta}_\delta^j(q',\omega')\rangle = \delta^{ij}\delta_{\gamma\delta}\delta^d(q+q')\delta(\omega+\omega'). \quad (17)$$

The noise term can be written as a conserved noise field $\zeta_A(x,t) \equiv \nabla \cdot \sqrt{2c_A^0 T_A}\,\eta_A(x,t)$, which has real- and Fourier-space correlation

$$\langle \zeta_\gamma(x,t)\zeta_\delta(x',t')\rangle = 2T_\gamma c_\gamma^0 \delta_{\gamma\delta}\nabla_x\nabla_{x'}\delta^d(x-x')\delta(t-t')$$
$$\langle \tilde{\zeta}_\gamma(q,\omega)\tilde{\zeta}_\delta(q',\omega')\rangle = 2T_\gamma c_\gamma^0 q^2 \delta_{\gamma\delta}(2\pi)^{d+1}\delta^d(q+q')\delta(\omega+\omega'). \quad (18)$$

The solution can immediately be found, to linear order, in terms of the noise by inverting the interaction matrix between fields. First put the dynamics in Fourier space. We find the equation

$$i\omega\tilde{\phi}_A(q,\omega) = c_A^0 q^2 \sum_\gamma (t_{A\gamma} - q^2 K_{A\gamma})\tilde{\phi}_\gamma(q,\omega) + \zeta_A(q,\omega) \quad (19)$$

whose solution can be written as

$$\begin{pmatrix} \tilde{\phi}_A(q,\omega) \\ \tilde{\phi}_B(q,\omega) \end{pmatrix} \equiv G_0(q,\omega) \begin{pmatrix} \zeta_A(q,\omega) \\ \zeta_B(q,\omega) \end{pmatrix}, \quad (20)$$

where we have defined the 0th-order matrix propagator $G_0(q,\omega)$.

One can immediately compute a number of quantities using the noise correlations (Eq. 18). For instance, the correlation between two fields' Fourier components is

$$\begin{aligned} &\langle \tilde{\phi}_\gamma(q,\omega)\tilde{\phi}_\delta(q',\omega')\rangle \\ &\quad = (2\pi)^{d+1}\delta^d(q+q')\delta(\omega+\omega') \\ &\qquad \cdot 2q^2 \left[G_0(q,\omega)[T\rho^0]G_0(q,\omega)^\dagger \right]_{\gamma\delta} \\ &\quad \equiv (2\pi)^{d+1}\delta^d(q+q')\delta(\omega+\omega')C_0(q,\omega)_{\gamma\delta} \end{aligned}$$

where $[T\rho^0]$ is the diagonal matrix with entries $T_A\rho_A^0$ and $\dagger$ is the Hermitian conjugate. By explicitly performing the matrix multiplication, we can then compute the static structure factors $S_0(q)_{\gamma\delta} = \int \frac{d\omega}{2\pi} C_0(q,\omega)_{\gamma\delta}$

$$\begin{aligned} S_0(q)_{AA} &= \frac{T_A\rho_A^0}{d_{AA}+d_{BB}} + \frac{T_A\rho_A^0 d_{BB}^2 + T_B\rho_B^0 d_{BA}^2}{(d_{AA}d_{BB}-d_{AB}d_{BA})(d_{AA}+d_{BB})} \\ S_0(q)_{BB} &= \frac{T_B\rho_B^0}{d_{AA}+d_{BB}} + \frac{T_A\rho_A^0 d_{AB}^2 + T_B\rho_B^0 d_{AA}^2}{(d_{AA}d_{BB}-d_{AB}d_{BA})(d_{AA}+d_{BB})} \\ S_0(q)_{AB} &= -\frac{T_A\rho_A^0 d_{AB}d_{BB} + T_B\rho_B^0 d_{BA}d_{AA}}{(d_{AA}d_{BB}-d_{AB}d_{BA})(d_{AA}+d_{BB})} \\ &= S_0(q)_{BA} \end{aligned} \quad (21)$$

where the d_{AA}, etc. are defined as

$$\begin{aligned} d_{AA} &= c_A^0(t_{AA} - q^2K_{AA}) \\ d_{AB} &= c_A^0(t_{AB} - q^2K_{AB}) \\ d_{BA} &= c_B^0(t_{AB} - q^2K_{AB}) \\ d_{BB} &= c_B^0(t_{BB} - q^2K_{BB}). \end{aligned} \quad (22)$$

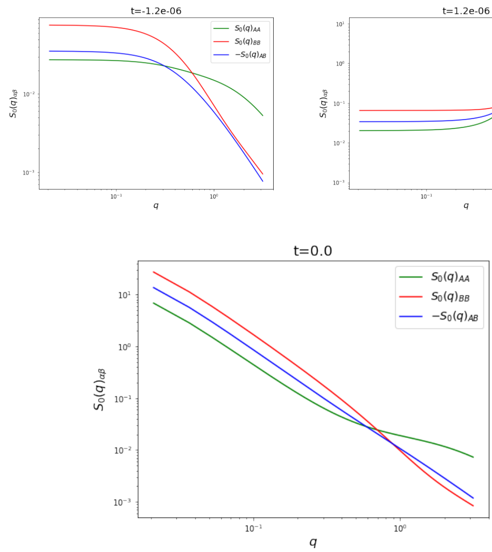

FIG. 5. Static structure factors at densities below (top left), above (top right), and at (bottom) the critical point. Note the characteristic power-law $S_0(q) \sim q^{-2}$ at the critical point, and the spike at a characteristic wavelength (indicating a characteristic clustering size) above the critical point.

See Fig. 5 for plots of this analytically computed structure factor near the critical point. With zero coupling ($t_{AB} = K_{AB} = 0$), the structure factors are $S_0(q)_{A\beta} = \delta_{A\beta} T_A c_A^0/(t_A + q^2 K_A)$, as is expected for a vanilla Gaussian model.

This concludes our discussion of the effective equilibrium theory of the two-temperature model.

III. DETAILED BALANCE VIOLATION

In this section, we ask: what happened to the violation of thermal equilibrium in this system? Evidently, it disappeared at some point during coarse-graining. Recovering it will require zooming back in and examining lost degrees of freedom.

A. Microscopic fluxes

First, return to the microscopics dynamics (Eq. 1b). In [7], the dynamics for a single pair of hot and cold particles ($N_A = N_B = 1$) with only an interaction potential $u^{AB}(r)$ was solved exactly. The relative coordinate between the two particles follows a Boltzmann distribution. That is,

$$P(r) = \frac{1}{z} \exp\left(-\frac{u_{AB}(r)}{T_{AB}}\right) \tag{23}$$

with effective temperature T_{AB} that is simply the arithmetic mean of T_A and T_B (Eq. 3).

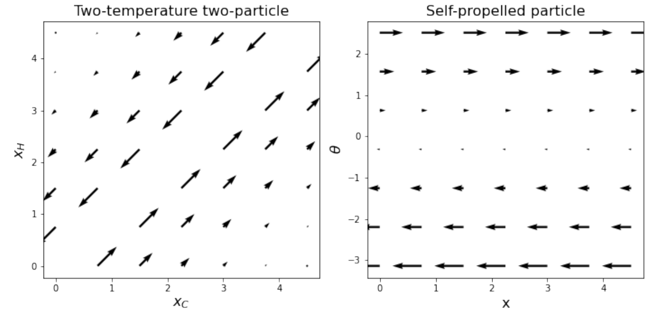

FIG. 6. Fluxes in configurational space for two coupled particles of different temperatures (left) and a single self-propelled particle (right).

From this, it was proven that there exist currents in the joint phase space of the hot and cold particles' locations. The hot particle tends to "push" the cold particle, resulting in an average current vector pointing in the direction of $\vec{x}_C - \vec{x}_H$. One can make an analogy to the microscopic currents in the configurational space of a single self-propelled particle, rotated by 45 degrees: the current vector points in the θ-direction in x-space. See Fig. 6. It is known that the introduction of an asymmetric potential causes directed currents in systems of self-propelled particles [5]. Given this similarity, it is natural to wonder whether directed currents will occur in a two-particle system with different temperatures.

We have observed such currents in these two-temperature systems. An example is shown in Fig. 7. The particles always tend to cross over the shorter, steeper side of the potential (this differs from the active particle ratchet current mechanism [5]).

B. Mesoscopic fluxes

To derive the effective free energy (Eq. 5), 3-body interactions were neglected in Ref. [7]. While such assumption is valid in the dilute limit, it becomes invalid when inter-particle interactions are mediated by the presence of another particle, i.e. by depletion forces. In Section VI of Ref. [8], they compute the effect of 3-body interactions on the equation of motion in the case of hard sphere interactions. The result is an additional term which can't be written as the gradient of a chemical potential, and therefore can't be described by an effective equilibrium theory. The equation of motion is

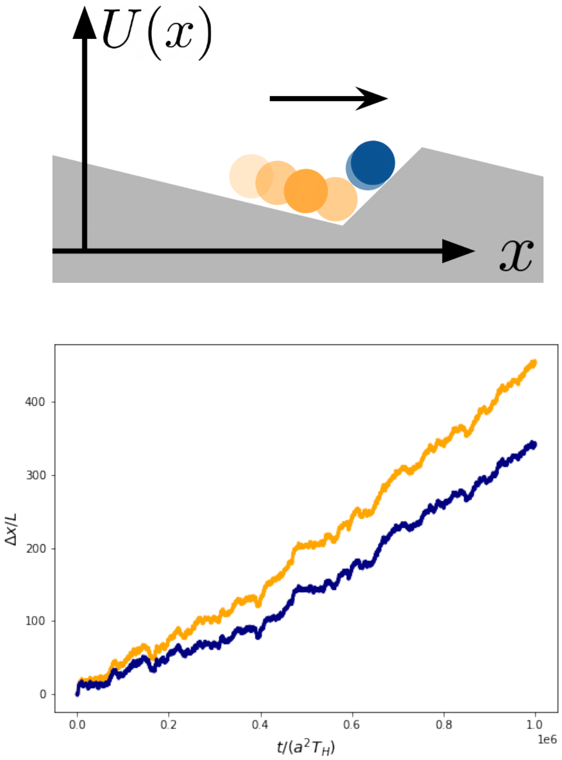

FIG. 7. Ratchet current in a system of 1 hot and 1 cold particle in a periodic, asymmetric external potential. Above: schematic of the ratchet mechanism in the type of potential landscape used. The hot particle "pushes" the cold particle over the hill. Below: net displacements of the particles over time.

$$\begin{aligned}\frac{\partial c_A}{\partial t} &= \nabla\left[c_A\left\{\nabla\mu_A + (T_A - T_B)c_A\nabla c_B\right\}\right]\\ \mu_A &= T_A\ln c_A + \frac{1}{2}T_A B_A c_A + T_{AB}B_{AB}c_B\\ &\quad + \frac{1}{2}T_A\Lambda_A\nabla^2 c_A + T_{AB}\Lambda_{AB}\nabla^2 c_B\\ &\quad + \frac{1}{2}T_A c_A^3 + \frac{1}{2}T_B c_B^3 + (T_A + 2T_B)c_A c_B \end{aligned} \tag{24}$$

In an equivalent "top-down" model where one would add the appropriate noise and assume roughly constant mobility (Eq. 15), the dynamics would then look like

$$\begin{aligned}\frac{\partial\phi_A}{\partial t} &= \nabla\left[c_A^0\left\{\nabla\left(\frac{\delta\mathcal{F}}{\delta\phi_A} + (T_A + 2T_B)\phi_A\phi_B\right)\right.\right.\\ &\quad \left.\left. + (T_A - T_B)\phi_A\nabla\phi_B + \sqrt{2c_A^0 T_A}\eta_A\right.\right] \end{aligned} \tag{25}$$

where $\mathcal{F}$ is an appropriate Landau-Ginzburg free energy with coefficients that can be determined by expanding the dynamics in eq. 24 around reference densities c_A^0, c_B^0.

In the case that $T_A \neq T_B$, there arises a term in the current of A and B particles, $\vec{J}_A = \vec{J}_B$, that cannot be expressed as the gradient of any potential. This current, $(T_A - T_B)\phi_A\nabla\phi_B$, is a direct consequence of the asymmetry of interactions between the particles of two temperatures: if one imagines a positive gradient of cold particles ($\nabla\phi_B$), the hot particles will have a current in that direction, i.e. they will "run up the hill," chasing the cold particles away.

Moreover, the curl of such current

$$\nabla\times\vec{J}_A = (T_A - T_B)\nabla\phi_A\times\nabla\phi_B \tag{26}$$

is generically nonzero in greater than 1 dimension.

By following the derivation of the entropy production in Ref. [12], one can verify that the entropy production, which comes solely from this new term, is nonzero, but only when there is a phase interface (just as in MIPS).

IV. CONCLUSION

We have investigated the phase separation in a system of particles coupled to two different thermal reservoirs. We have found decent agreement between simulations of Ref. [9] and this work with the effective equilibrium description of the system. We have also shown that there are currents that break detailed balance at the micro- and meso-scopic levels. In particular, we have observed a ratchet current in a two-temperature two-particle system in an asymmetric potential due to the "chasing" dynamics of the particles, and traced these chasing dynamics to the mesoscopic level where they add terms (derived in Ref. [8]) to the current that can't be described by an effective free energy.

However, the question remains whether or not this mesoscopic detailed balance violation will have any macroscopic consequences on the phase. One could proceed with a renormalization group analysis to check its relevance. Dimensional analysis indeed suggests that in the long-wavelength limit, these terms are relevant compared to the linear model, at least below the upper critical dimension $d_c = 4$. If we apply the scaling $x \mapsto bx$, $\phi \mapsto b^\alpha\phi$, and $t \mapsto b^z t$, the dynamics (Eq. 25) become

$$\begin{aligned}\frac{\partial\phi_A}{\partial t} &= \nabla\left[c_A^0\left\{\nabla\sum_\gamma\left(b^{z-2}t_{A\gamma}\phi_{A\gamma} + b^{z-4}K_{A\gamma}\nabla^2\phi_\gamma\right)\right.\right.\\ &\quad \left. + b^{z+\alpha-2}(T_A - T_B)\phi_A\nabla\phi_B\right\}\\ &\quad \left. + b^{-d/2+z/2-\alpha-1}\sqrt{2c_A^0 T_A}\eta_A\right]. \end{aligned} \tag{27}$$

Selecting exponents that fix the fluctuations $K_{A\gamma}$ and noise T_A ($z = 4, \alpha = (2-d)/2$), we find

$$(T_A - T_B)\phi_A\nabla\phi_B \mapsto b^{(4-d)/2}(T_A - T_B)\phi_A\nabla\phi_B, \tag{28}$$

signifying a relevant coupling.

Moreover, we haven't yet confirmed what will happen to the phase separation of many particles in the presence of an asymmetric potential (like in Fig. 7). Random potentials have been shown to destroy phase separation in active particle suspensions via the creation of ratchet currents [15]. Because a pair of hot and cold particles exhibit a ratchet current in an asymmetric potential, it is natural to ask whether such ratchet currents will emerge and have a macroscopic effect on the phase separation.

In future work, we hope to address these questions, in addition to generalizing our theory to systems with many different temperatures. This will help us better understand the effects of inhomogenous activity in biological systems such as active polymers, where temperature-based phase separation is already known to occur [16].

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