Superradiant Phase Transitions in Quantum Optics

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Phase transitions in quantum optics and photonics are valuable from both a fundamental and applied perspective. On the fundamental side, they can help unlock a better understanding of many-body quantum systems interacting through QED. However, phase transitions underlie the operation of numerous photonic technologies, including lasers, optical parametric oscillators, and next-generation sensors and energy harvesting devices that operate on the principle of superradiance. Here, we explore phase transitions in Dicke superradiance as well as laser thresholds, and begin developing the first formalism for modeling Smith-Purcell (SP) superradiance as a phase transition. This work could comprise the first theory to date for understanding the nature of the SP superradiant phase transition, paving the way towards engineering novel free electron devices such as nanoscale free electron lasers.

CONTENTS

I. INTRODUCTION

Phase transitions are ubiquitous in physics and they have enabled explanations of some of the most intriguing behaviors in photonic devices, from superradiance in ensembles of quantum emitters to phase transitions below and above threshold in lasers and optical parametric oscillators [1, 2].

In this paper, we discuss the application of statistical phase transition theory to two phenomena in photonics:

1. Quantum optical superradiance. The most famous example of this class of phenomena is Dicke superradiance, characterized by coherent radiation from N emitters whose intensity is enhanced by a factor of N relative to the incoherent case. The onset of Dicke superradiance can be modeled as a disorder-to-order phase transition controlled by temperature and coupling strength between the emitters and an electromagnetic mode.

2. Laser thresholds. The behavior of lasers below and above threshold (which demarcates when lasing actually occurs) will be shown to be very similar to second-order phase transitions in ferromagnetic spin systems.

Lastly, we begin developing for the first time a model for superradiant Smith-Purcell radiation that attempts to construct a Landau-Ginzburg type Hamiltonian for the free electron-electromagnetic mode interaction. This Hamiltonian is built from the classical dynamics governing the interaction between free electrons and an electromagnetic wave [3, 4]. Just as with mode locking in lasers, a disorder to order phase transition occurs that selects a single mode that dominates SP radiation. The critical point we seek is a discontinuity in the power law dependence of emitted power with electron current, or a discontinuity in the so-called "bunching factor," either (or both) of which should characterize the order parameter.

II. SUPERRADIANT PHASE TRANSITIONS IN THE DICKE MODEL

We begin by considering the thermodynamic limit of the Dicke model, which in the simplest case describes the dipolar interaction between a collection of N atoms (two-level systems) and a single electromagnetic field mode at angular frequency ω [5]. The Hamiltonian can be written

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as ($\hbar = 1$)

$$\mathcal{H}_{\text{Dicke}} = \omega a^\dagger a + \sum_{i=1}^{N} \left(\Delta\sigma_{iz} + \frac{g}{\sqrt{N}}\sigma_{ix}\left(a + a^\dagger\right)\right), \quad (1)$$

where the first term is the bare Hamiltonian of the electromagnetic (EM) field ($a, a^\dagger$ are the usual raising and lowering operators satisfying the canonical commutation relation), the second is the bare Hamiltonian of the atoms, and the last term is a dipolar coupling between the atoms and the field. Note that $\sigma_{ix,iz}$ are the Pauli matrices for atom i, so that each atom has an energy splitting between ground and excited states of 2Δ. The N dependence of the coupling has been removed from g. Specifically, the $\sqrt{N}$ normalization will become clearer below, but can roughly be seen by considering a mean field substitution $a^{(\dagger)} \to \alpha^{(*)}$. For superradiant phenomena, we are concerned about the ratio $\langle a^\dagger a\rangle/N$, which should be finite in SP. Then, $\alpha \sim \sqrt{N}$, canceling the explicit N dependence in the coupling term inside the summation.

II.1. Order Parameter

We expect the order parameter to scale as $\mathcal{O} \sim \langle a^\dagger a\rangle/N$ such that $\mathcal{O} \to 0$ in the disordered phase (no superradiance) and $\mathcal{O}$ tends to a nonzero finite value in the ordered phase (superradiance). With these considerations, we consider the ansatz $\mathcal{O} = \langle a^\dagger a\rangle/(CN)$. Keeping the mean field approach in mind where we treat $a, a^\dagger$ like complex numbers, the bare photon Hamiltonian scales as $CN\omega\mathcal{O}$ while the interaction Hamiltonian scales as $Ng\sqrt{C\mathcal{O}}$ and the bare atomic (qubit) Hamiltonian scales as $N\Delta$. Experiments indicate that, in addition to the thermodynamic limit $N \to \infty$, superradiant phase transitions are also observable in the "classical oscillator" limit, $\omega/\Delta \ll 1$. Thus we posit $C = \Delta/\omega$, so that

$$\mathcal{O} = \omega\langle a^\dagger a\rangle/(\Delta N) \quad (2)$$

This ensures the order parameter is zero in the disordered phase in the thermodynamic and classical oscillator limits.

II.2. Mean Field Approach

We now try computing observables in the mean field theory (MFT) limit by taking $a \to \alpha, a^\dagger \to \alpha^*$ (essentially assuming the EM field behaves like a strong coherent state), in which case the atomic-dependent part of the Hamiltonian reads (for a single atom coupled to the EM mode)

$$\mathcal{H}_{\text{MF}} = \begin{cases} \begin{pmatrix} \Delta & g\alpha/\sqrt{N} \\ g\alpha^*/\sqrt{N} & -\Delta \end{pmatrix} & \text{(RWA)} \\ \begin{pmatrix} \Delta & 2\text{Re}(\alpha)g/\sqrt{N} \\ 2\text{Re}(\alpha)g/\sqrt{N} & -\Delta \end{pmatrix} & \text{(no RWA).} \end{cases} \quad (3)$$

Here, RWA denotes the "rotating wave approximation," which discards quickly-rotating terms in the dipole interaction Hamiltonian so that

$$\sigma_x\left(a + a^\dagger\right) = \left(\sigma^+ + \sigma^-\right)\left(a + a^\dagger\right) \quad (4)$$
$$\approx a\sigma^+ + a^\dagger\sigma^-, \quad (5)$$

where $\sigma^\pm$ denote the atomic raising and lowering operators. Thus, in the RWA, we only retain "energy-conserving" terms where one creates a photon and de-excites the atom or annihilates a photon and excites the atom. The RWA is convenient for weak couplings, but can break down for strong couplings g. In what directly follows, we do not use the RWA, but an examination of the Tavis-Cummings model, where the RWA is assumed, is provided in Sec. II.3.

We would like to compute the partition function $Z = \text{Tr}\left(e^{-\beta\mathcal{H}}\right)$. Tracing over the atomic part is simple following a transfer matrix-like approach on $\mathcal{H}_{\text{MF}}$. Diagonalizing $\mathcal{H}_{\text{MF}}$ gives eigenvalues $\lambda_\pm = \pm\sqrt{\Delta^2 + 4g^2x^2/N}$, where $x \equiv \text{Re}(\alpha)$. We also let $y \equiv \text{Im}(\alpha)$, so that $|\alpha|^2 = x^2 + y^2$. Then, the partition function for the Dicke model with N atoms reads

$$Z_{\text{MF}} = \int \frac{dxdy}{\pi} \exp\left[-\phi(x, y)\right]$$
$$\phi(x, y) = \beta\omega(x^2 + y^2) - N\ln\left(2\cosh\left(\beta\sqrt{\Delta^2 + 4g^2x^2/N}\right)\right) \quad (6)$$

To compute the partition function, we just need to integrate over the overcomplete basis of coherent states $|\alpha\rangle$ as indicated above. Performing the Gaussian integral over y, we find

$$Z_{\text{MF}} = \frac{1}{\sqrt{\pi\beta\omega}} \int dx \exp\left[-\phi(x, 0)\right] \quad (7)$$

Let us now attempt to use the saddle point method to find the order parameter $\mathcal{O}$ in the ordered and disordered phases. Note here that $\mathcal{O} = \omega|\alpha|^2/(N\Delta) \approx \omega x^2/(N\Delta)$, so that

$$\phi(x, 0) = N\left(\beta\Delta\mathcal{O} - \ln\left(2\cosh\left(\beta\sqrt{\Delta^2 + 4g^2\Delta\mathcal{O}/\omega}\right)\right)\right) \quad (8)$$

Differentiating with respect to $\mathcal{O}$ shows that the saddle point solutions satisfy

$$\frac{\omega f(\mathcal{O})}{2g^2} - \tanh(\beta f(\mathcal{O})) = 0, \quad (9)$$

where $f(\mathcal{O}) = \sqrt{\Delta^2 + 4g^2\Delta\mathcal{O}/\omega}$. The phase diagram can be found by setting $\mathcal{O} = 0$ in the disordered phase, so that the phase transition boundary is

$$\frac{\omega\Delta}{2g_c^2} - \tanh(\beta\Delta) = 0 \quad (10)$$

Notice that close to criticality, a Taylor expansion of Eq. 10 about $g \to g_c, \mathcal{O} \to 0$ reveals that $\mathcal{O} \propto |g - g_c|$.

We provide a plot of the phase diagram in the parameter space g, T in Fig. 1. Note that $g > g_c$ corresponds to the superradiant phase. Note that superradiance does not occur spontaneously at zero temperature - the required coupling strengths are $g > \sqrt{\omega\Delta/2}$.

II.3. Generalized Dicke Model and Tavis-Cummings Model

We generalize the Dicke model presented in Eq. 16 to include atom-dependent coupling and bare atomic energy:

$$\mathcal{H} = \omega a^\dagger a + \sum_{i=1}^{N} \left(\Delta_i \sigma_{iz} + \frac{g_i}{\sqrt{N}} \sigma_{ix} \left(a + a^\dagger\right) \right), \quad (11)$$

in which case the condition for superradiance $\sum_i (g_i/g_{ci})^2 > 1$ becomes

$$\sum_{i=1}^{N} \tanh(\beta\Delta_i) \frac{g_i^2}{\omega\Delta_i} > N/2 \quad (12)$$

When the RWA applies, we find a partition function of the form

$$\begin{aligned} Z_{\text{TC}} &= \int \frac{dxdy}{\pi} \exp\left[-\phi(x,y)\right] \\ \phi(x,y) &= \beta\omega(x^2+y^2) - \\ &\sum_i \ln\left(2\cosh\left(\beta\sqrt{\Delta_i^2 + g_i^2(x^2+y^2)/N}\right)\right) \end{aligned} \quad (13)$$

We can convert to polar coordinates and integrate over phase, yielding a single integral over $r \equiv |\alpha|$. Note that the potential $\phi(x,y)$ only depends on $|\alpha| = \sqrt{x^2+y^2}$ and resembles a Landau-Ginzburg type potential (e.g., for low temperature and weak coupling). Superradiance occurs for

$$\sum_{i=1}^{N} \tanh(\beta\Delta_i) \frac{2g_i^2}{\omega\Delta_i} > 2N \quad (14)$$

When this condition is satisfied, spontaneous symmetry breaking of the Tavis-Cummings model's global U(1) symmetry occurs, with a transition from zero to nonzero $\mathcal{O}$ accompanied by generation of a Goldstone mode.

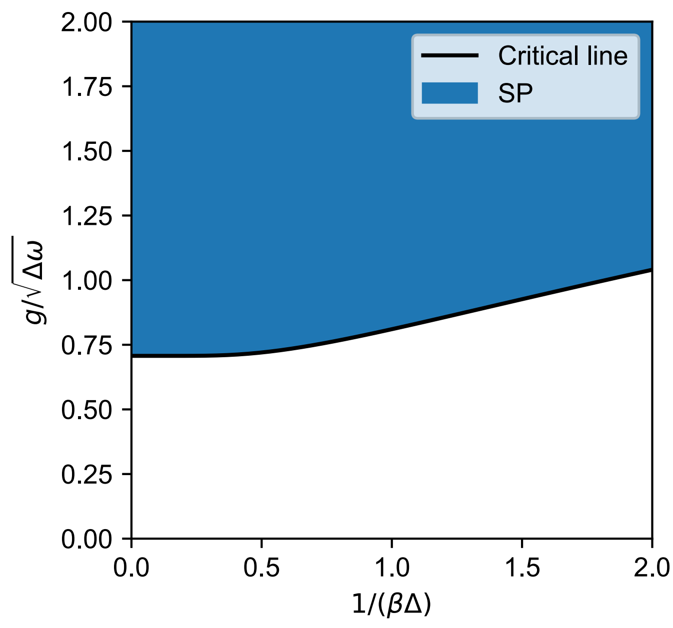

FIG. 1: Superradiance phase diagram for the basic Dicke model in the parameter space of temperature T and coupling strength g.

II.4. The Role of Symmetry

The second-order superradiant phase transition just described is associated with a broken symmetry. For example, by introducing a bias term to the Dicke Hamiltonian, $\mathcal{H} = \mathcal{H}_{\text{Dicke}} + \sum_{i=1}^{N} e_i \sigma_{ix}$, we break the Z_2 symmetry explicit in the Landau potential in Eq. 8. Instead, the Landau potential now reads

$$\begin{aligned} \phi(x,y) &= \beta\omega(x^2+y^2) \\ &- N \ln\left(2\cosh\left(\beta\sqrt{\Delta^2 + \left(2gx/\sqrt{N} + e\right)^2}\right)\right) \end{aligned} \quad (15)$$

where for simplicity we took $e_i = e, \forall i$. Note that $x = 0$ is no longer a saddle point solution for this Landau potential since $\phi(x,0)$ is not an even function of x. The order parameter $\mathcal{O}$ changes continuously. In the case where the biases e_i are different, the phase transition is continuous if at least one of the biases is nonzero.

Breaking a symmetry is not necessary for a first-order transition. Consider the Dicke model supplemented by two-photon coupling terms

$$\begin{aligned} \mathcal{H} &= \omega a^\dagger a \\ &+ \sum_{i=1}^{N} \Delta\sigma_{iz} + \left(\frac{g}{\sqrt{N}}\left(a + a^\dagger\right) + \frac{g'}{N}\left(a^2 + a^{\dagger 2}\right)\right)\sigma_{ix}, \end{aligned} \quad (16)$$

The introduction of these two-photon terms necessitates constraints on the parameters in the Hamiltonian to guarantee stability. That is, in order to have Boltzmann weights that do not diverge with the size of the coherent state (essentially trapping it in an attractive potential), we demand $g'(x^2+y^2) + \omega xy > 0$ by looking at the quadratic terms in the Hamiltonian. This yields $g'/\omega < 1/2$ as the stability condition.

We now compute the Landau potential as

$$\begin{aligned} \phi(x,y) &= \beta\omega x^2 \\ &- N \ln\left(2\cosh\left(\beta\sqrt{\Delta^2 + \left(\frac{2gx}{\sqrt{N}} + \frac{2g'(x^2-y^2)}{N}\right)^2}\right)\right). \end{aligned} \quad (17)$$

This clearly lacks any symmetry but nonetheless still has $\mathcal{O} = 0$ as a saddle point solution since the potential can be expanded as a power series in $\mathcal{O}$ with no constant term when g, g' are small. The nonzero saddle point extrema can be found via $\partial\phi/\partial x = \partial\phi/\partial y = 0$. In this simultaneous system of equations, $y = 0$ is constrained by stability, so we need only optimize $\phi(x, 0)$. The critical line can then be found via $\phi(x, 0) = \phi(0, 0)$ and $\phi'(x, 0) = 0$. For the limit $\beta\Delta \to \infty$ (e.g., at low temperature), the critical line reads

$$2\gamma_c^2 + 4\gamma_c'^2 = 1, \tag{18}$$

where $\gamma = g/\sqrt{\Delta\omega}, \gamma' = g'/\omega$.

II.5. Validity of Mean Field Theory

Here, we consider the validity of mean field theory. For concreteness, we consider the original Dicke model presented at the start of this paper. Earlier work [6] has shown that the true partition function can be bounded by

$$Z_{\mathrm{MFT}} \le Z \le \exp{(\beta\omega)} Z_{\mathrm{MFT}}. \tag{19}$$

We can rearrange this to read

$$-\frac{\ln Z_{\mathrm{MFT}}}{\beta N\Delta} \ge -\frac{\ln Z}{\beta N\Delta} \ge -\frac{\omega}{N\Delta} - \frac{\ln Z_{\mathrm{MFT}}}{\beta N\Delta}. \tag{20}$$

If we were to complete the saddle point integral in Eq. 7, we would find in the limit $N\Delta\omega \to \infty$

$$Z_{\mathrm{MF}} \propto \frac{1}{\beta\omega} \exp{(-\beta N\Delta\alpha)}, \tag{21}$$

where α is an unknown constant related to ϕ'' evaluated at the saddle point solution. Thus $\frac{\ln Z_{\mathrm{MFT}}}{\beta N\Delta}$ remains finite in the limit $N\Delta\omega \to \infty$, so that the mean field solution is the correct approximation to the true partition function in this limit.

III. PHASE TRANSITIONS IN LASERS

III.1. Laser Threshold

In a conventional laser, the lasing steady state occurs when gain balances loss. The rate equation for photon number generally takes the form

$$\langle\dot{n}\rangle = (G(S, n) - \kappa)n + F_n, \tag{22}$$

where $G(S, n)$ denotes the gain, κ the loss, and F_n a Langevin force term satisfying $\langle F_n^\dagger F_n\rangle = 2\kappa n$. The gain is of course proportional to the population inversion S but also depends on n due to gain saturation. The gain generally saturates according to $G \to G/(1+n/n_s)$ where n_s is the saturation photon number. If we look at the n dependence, and being careful about expectation values,

$$\langle\dot{n}\rangle = (G - \kappa)\langle n\rangle - \frac{G}{n_s}\langle n^2\rangle + G + F_n, \tag{23}$$

where the constant term G_0 is the spontaneous emission rate into the laser mode. For computing electric fields, where we expect to see a phase transition below and above threshold, it is most convenient to work in the basis of coherent states $|\alpha\rangle$, which are eigenstates of the annihilation operator: $a\,|\alpha\rangle = \alpha\alpha$. The photon field state is given by a density matrix

$$\rho = \int d^2\alpha P(\alpha, \alpha^*, t)\,|\alpha\rangle\,\langle\alpha|\,. \tag{24}$$

P obeys the Fokker-Planck equation

$$\begin{aligned}\frac{\partial P}{\partial t} =& -\frac{\partial}{\partial\alpha}\left(\frac{1}{2}(G-\kappa)\alpha P - \frac{G}{2n_s}|\alpha|^2\alpha P\right) \\ &- \frac{\partial}{\partial\alpha^*}\left(\frac{1}{2}(G-\kappa)\alpha^* P - \frac{G}{2n_s}|\alpha|^2\alpha^* P\right) \\ &+ G\frac{\partial^2 P}{\partial\alpha\partial\alpha^*}\end{aligned} \tag{25}$$

where the first two terms respectively represent net amplification of the field and the last term gives rise to laser linewidth via diffusion of the electric field. The steady state solution of this Fokker-Planck equation is

$$P(\alpha, \alpha^*) \propto \exp\frac{\frac{1}{4}(G-\kappa)|\alpha|^2 - \frac{G}{8n_s}|\alpha|^4}{G/4}, \tag{26}$$

which on computing expectation values for the electric field $E \propto \frac{1}{2}\left(a + a^\dagger\right)$ gives the equation of motion for electric field

$$\langle\dot{E}\rangle = \frac{1}{2}(G-\kappa)\langle E\rangle - \frac{G}{2n_s}\langle E^3\rangle + \eta(t). \tag{27}$$

This looks like a time-dependent Landau-Ginzburg equation with effective potential $V(E) \propto -\frac{1}{4}(G-\kappa)E^2 + \frac{G}{8n_s}E^4$ and the equilibrium probability distribution $P(\alpha, \alpha^*)$ is Boltzmann. In a mean field approach, we take $\langle E^3\rangle \approx \langle E\rangle^3$ so that the steady state solutions are

$$\langle E\rangle = \begin{cases} 0 & \Delta S < 0 \\ \sqrt{\frac{G_S\Delta S}{SG_S/n_s}} & \Delta S > 0 \end{cases} \tag{28}$$

where $\Delta S \equiv S - S_t$ denotes the population inversion above the threshold S_t and $G \equiv SG_S, \kappa \equiv S_t G_S$. Here $\langle E\rangle$ just refers to the magnitude of the electric field: note that, just like the case of ferromagnetic spins (without an external magnetic field), the average electric field is really zero due to all phases having equal weight.

When an external driving field Λ (injected field) is present, this phase symmetry is broken, so $\langle\vec{E}\rangle \ne 0$. In this case, the extremum of the Landau potential is found via

$$\frac{1}{2}(G-\kappa)\langle E\rangle - \frac{G}{2n_s}\langle E^3\rangle + \Lambda = 0. \tag{29}$$

This looks just like ferromagnetic spins in a magnetic field and the critical behavior is analogous.

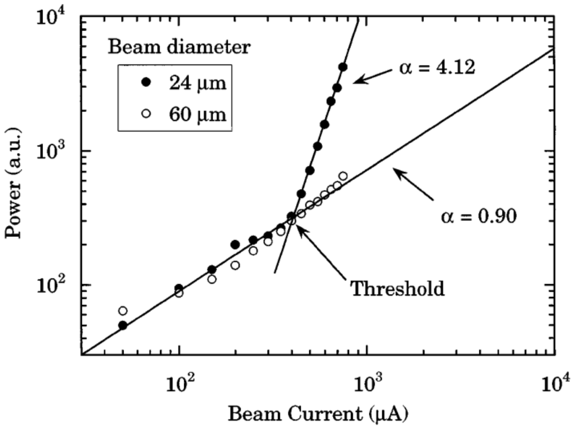

FIG. 2: Smith-Purcell superradiant phase transition. At some threshold electric current, the power law dependence of emitted radiation on current changes drastically from the incoherent to coherent regimes [7].

IV. SMITH-PURCELL SUPERRADIANT PHASE TRANSITIONS

In this section, we begin developing the first model of Smith-Purcell (SP) superradiance as a phase transition. SP superradiance can occur when periodically prebunched electrons pass near a nanophotonic grading, and its frequency is a harmonic of the bunching frequency. Two regimes of SP superradiance have been found: self-amplified (no pre-bunching, bunching occurs through positive feedback between an electron beam and evanescent mode) and stimulated (electrons are pre-bunched via optical modulation before interacting with the grating). In either case, superradiance emerges because the electromagnetic fields act back on the electrons, enhancing bunching and coherent buildup of radiation.

SP superradiance is exciting both from a fundamental physics point of view as well as from a practical standpoint, where the radiation it produces could be harnessed for next-generation nanoscale free electron lasers (FEL) that provide coherent light across an unprecedented range of the electromagnetic spectrum. Modeling self-amplified SP superradiance as a phase transition has not been investigated before and may help elucidate the nature of the critical point marking onset of SP superradiance. This critical point has been probed in experiment (Fig. 2) and simulation but is not well understood theoretically [7]. For example, a generalized onset parameter of superradiance that accurately estimates the threshold for FEL in the self-amplified regime is lacking.

IV.1. Ignoring Back-Action of Electron-Photon Dynamics

We consider a Hamiltonian that is the sum of bare electron and photon Hamiltonians as well as a coupling $\mathbf{J}\cdot\mathbf{E}$ that represents the rate at which the electromagnetic fields do work on the charges. Integrating this last term over time, interaction length, and cross section adds a potential energy to the electron Hamiltonian (classically, this looks like $-e\int \mathbf{E}\cdot\mathbf{ds}$). Using a mean field approach where we assume the fields are constant over the interaction region (i.e., neglecting the effect of the electrons back on the field), we write (taking $\hbar = c = e = m_e = 1$)

$$\mathcal{H}_{\rm SP} = \omega a^\dagger a + \sum_{i=1}^N \gamma_i + L\sqrt{\frac{\omega}{V}} i\left(ae^{-i\chi_i} - a^\dagger e^{i\chi_i}\right), \quad (30)$$

where γ_i denotes the Lorentz factor of electron i (recall that the total relativistic energy is given by γmc^2), L denotes the interaction length and $\chi_i(z) = kz - \omega\tau_i(z)$, with $\tau_i(z) = \tau_i(0) + \int_0^z dz' 1/v_i(z')$ the time for electron i with velocity $v_i(z')$ to reach point z, denotes the phase matching between the electron and photon ($\chi_i = 0$ indicates the electron and photon are perfectly phase-matched). The interaction term is derived from [8]. In this simplest model, we emphasize again that we are neglecting the action of the electrons back on the field as well as the spatial dependence of $\chi_i(z)$ and the fields $a(z)$. The interpretation is that the mode acts as a strong driving field that causes the electrons to bunch. We write the "bunching factor" as

$$b(z) = \langle e^{-i\chi(z)}\rangle \quad (31)$$

though we will drop the z dependence below. If the electrons are "bunched" $b \neq 0$ whereas if the electrons are "unbunched" their phases are randomly distributed and $b = 0$.

To compute the partition function, we can integrate over the electron momenta as

$$Z_\gamma = \left(\int_0^\infty 4\pi p^2 dp e^{-\beta\sqrt{p^2+m^2}}\right)^N, \quad (32)$$

so that the total partition function reads

$$\begin{aligned} Z_{\rm SP} &= Z_\gamma \int \frac{dxdy}{\pi}\prod_i d\chi_i e^{-\beta\left(\omega(x^2+y^2)+g\sum_i(y\cos\chi_i - x\sin\chi_i)\right)} \\ &= \frac{1}{\beta\omega}\int\prod d\chi_i e^{\frac{\beta g^2}{4\omega}\left(\langle\cos\chi_i\rangle^2+\langle\sin\chi_i\rangle^2\right)} \end{aligned} \quad (33)$$

where as before $x \equiv \mathrm{Re}(\alpha), y \equiv \mathrm{Im}(\alpha)$.

With the partition function thus written down, we pause to consider the expected order parameter. Perhaps the bunching factor b is an order parameter, in which case

$$\begin{aligned} \langle\sin\chi_i\rangle^2 + \langle\cos\chi_i\rangle^2 &= \left(\frac{\mathcal{O}-\mathcal{O}^*}{2i}\right)^2 + \left(\frac{\mathcal{O}+\mathcal{O}^*}{2}\right)^2 \\ &= |\mathcal{O}|^2. \end{aligned} \quad (34)$$

Or, perhaps the order parameter looks like $\mathcal{O} = \langle a^\dagger a\rangle/N$ in the spirit of the experimental data in Fig. 2. In either case, we expect $\mathcal{O} = 0$ in the disordered phase (randomly phased electrons, incoherent emission) and $\mathcal{O} \neq 0$ in the SP phase. However, in the current formulation, neither of the two suspected order parameters show a discontinuous transition.

IV.2. Including Complete Electron-Photon Dynamics

We now try to construct a Hamiltonian from the equations of motion, completely including the electron-photon dynamics. In rescaled (dimensionless) variables, the equations of motion for the electron energy γ_i, phase χ_i, and field a are given by [8]:

$$\dot{a} = \alpha \langle e^{-\chi_i} \rangle \tag{35a}$$

$$\dot{\gamma}_i = -\frac{1}{2}\left(a e^{i\chi_i} + a^* e^{-i\chi_i}\right) \tag{35b}$$

$$\dot{\chi}_i = \Omega\left(\frac{1}{\beta_i} - \frac{1}{\beta_{\text{ph}}}\right), \tag{35c}$$

where $\gamma_i = 1/\sqrt{1-\beta_i^2}$, $\Omega = \omega d/c$ is a normalized angular frequency of the field, β_{ph} is the normalized phase velocity, and α is a coupling constant that is directly proportional to the current I and the interaction impedance Z, and also depends on the geometry of the interaction region. Before proceeding, we can appreciate the effect of superradiance by considering the power of emitted radiation. For a single electron, the variation in the electric field is $\frac{da_i}{dz} = \alpha e^{-i\chi_i(z)}$. Approximating the velocity v_i as constant over the interaction region (i.e. ignoring the effect of the radiation on the electron), we find that after the interaction region (of length d), the normalized electric field goes as

$$a_i(d) = \alpha d e^{-i\chi_i(0) - i(\omega/v_i - k)d/2} \text{sinc}\left((\omega/v_i - k)\frac{d}{2}\right). \tag{36}$$

Notice that for randomly phased electrons, the average electric field $\langle E \rangle \propto \langle a \rangle = 0$. However, the average power of spontaneous radiation $\propto \langle E^2 \rangle$ can be computed from the Poynting flux as

$$P_{\text{sp}} \propto I \text{sinc}^2\left((\omega/v_i - k)\frac{d}{2}\right) \tag{37}$$

so that the power scales as N. In the superradiant case, the energies γ_i and phases $\chi_i(0)$ are tightly bunched: suppose $v_i \approx v_0$ and $-\pi < -\chi_0 < \chi_i(0) < \chi_0 < \pi$. Then, the average electric field goes as

$$a(d) \propto \alpha \text{sinc}(\chi_0) e^{-i(\omega/v_i - k)d/2} \text{sinc}\left((\omega/v_i - k)\frac{d}{2}\right), \tag{38}$$

where $\text{sinc}(\chi_0)$ emerges from averaging over the phases $\chi_i(0)$. Notice $\chi_0 \to \pi$ recovers the incoherent case. Now, we immediately see that the power is superradiant,

$$P_{\text{sr}} \propto I^2 \text{sinc}^2(\chi_0) \text{sinc}^2\left((\omega/v_i - k)\frac{d}{2}\right), \tag{39}$$

scaling as N^2.

Returning to our Hamiltonian construction, we can show that energy conservation applies in the sense

$$\frac{d}{dz}\left(\langle \gamma_i \rangle + \frac{1}{2\alpha}|a|^2\right) = 0, \tag{40}$$

where z is dimensionless and $\dot{x} \equiv dx/dz$ notationally. Thus, $\langle \gamma_i \rangle + \frac{1}{2\alpha}|a|^2 = E_0$ for some constant E_0. This allows one to eliminate the fields from the equations of motion. Specifically, we can let

$$a(z) = A(\gamma_i(z)) e^{i\theta(z)} \tag{41}$$

$$A(\{\gamma_i(z)\}) = \sqrt{2\alpha\left(E_0 - \langle \gamma_i \rangle\right)} \tag{42}$$

$$q_i(z) = \chi_i(z) + \theta(z) \tag{43}$$

Thus, the equations of motion now read

$$\dot{\gamma}_i = -A(\{\gamma_i\}) \cos q_i \tag{44a}$$

$$\dot{q}_i = \Omega\left(\frac{1}{\sqrt{1 - 1/\gamma_i^2}} - \frac{1}{\beta_{\text{ph}}}\right) - \frac{\alpha}{A(\{\gamma_i\})}\langle \sin q_i \rangle \tag{44b}$$

We can treat these equations as Hamilton equations of motion for the following Hamiltonian:

$$\mathcal{H}(\{\gamma_i, q_i\}) = \sum_i \Omega\left(\sqrt{\gamma_i^2 - 1} - \frac{\gamma_i}{\beta_{\text{ph}}}\right) + A(\{\gamma_i\}) \sin q_i \tag{45}$$

with $\dot{q}_i = \partial\mathcal{H}/\partial\gamma_i, \dot{\gamma}_i = -\partial\mathcal{H}/\partial q_i$.

We now use Eq. 45 as the new starting point for understanding the superradiant phase transition. We can proceed in a few ways:

1. Applying a field theory-type approach to the electrons in addition to the EM mode, we have as before $a \to \alpha$ and $\chi_i \to \chi(\mathbf{x})$, $\gamma_i \to \gamma(\mathbf{x})$ with the bunching factor
$$b = \frac{1}{V}\int d^3\mathbf{x} e^{-i\chi(\mathbf{x})}. \tag{46}$$
2. Transforming Eq. 44 to Fokker-Planck/Landau-Ginzburg form, such as $\dot{a} = \partial\mathcal{V}/\partial a + \eta(t)$, where $\eta(t)$ is an effective temperature introduced by noise in the electron phases. Ideally, we would have a single nonlinear equation in the field variable [9] that could be solved using techniques such as those presented in [10].

These approaches will be explored more fully in future work. For the latter approach, one could find equations of motion for "macroscopic variables" rather than variables like γ_i, χ_i for individual electrons. These three macroscopic variables should include the field amplitude a and bunching factor b, and their equations of motion would replace Eq. 44. Because of the nonlinearity of this system, to obtain a closed set of differential equations describing the interaction, it becomes necessary to truncate the system of equations at some order using an approximation about the correlations between the microscopic variables [11]. For example, we can begin constructing

$$\begin{aligned} \dot{a} &= \alpha b \\ \dot{b} &= \frac{i\Omega}{\beta_{\text{ph}}} b - i\Omega\left\langle \frac{e^{-i\chi_i}}{\beta_i} \right\rangle, \end{aligned} \tag{47}$$

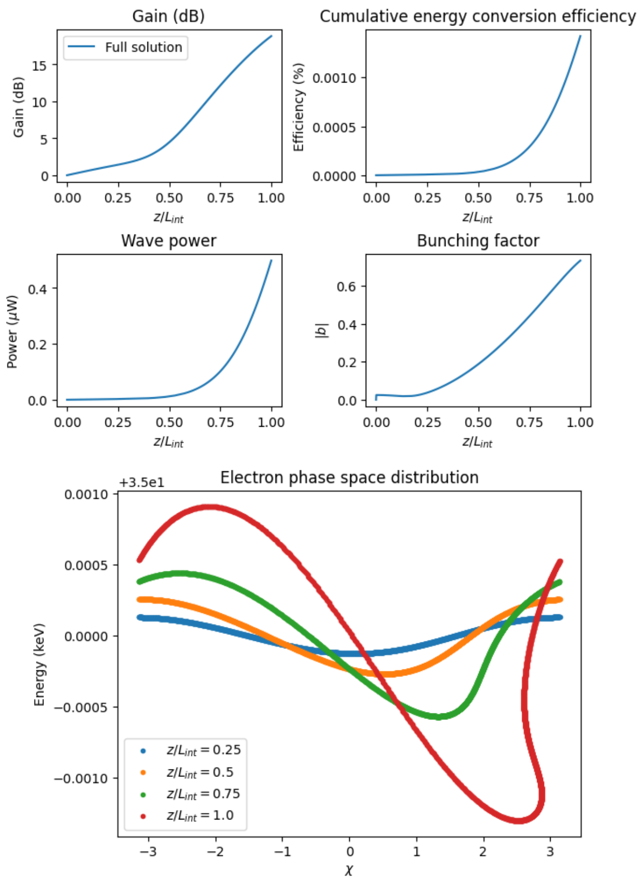

FIG. 3: Nonsuperradiant regime. Top: plots of the gain, energy conversion efficiency, mode power, and bunching factor across the interaction region. Bottom: electron bunching induced by the interaction, as manifested in a phase space plot.

where the term in angular brackets needs its own equation of motion, which in turn generates higher order terms. After truncating the system of equations, we can isolate a nonlinear differential equation for one of the macroscopic variables (e.g., a) and apply Landau-Ginzburg formalism.

Complementing this analysis, I aim to compare the results using direct numerical simulations of Eq. 44. Fig. 3 presents preliminary results of these simulations in the nonsuperradiant regime.

V. CONCLUSION

In this paper, we have provided a brief overview of the application of Landau-Ginzburg formalism to the analysis of quantum optical superradiant phase transitions. We have also begun developing the foundations for a phase transition theory of Smith-Purcell superradiance, drawing analogies with Dicke superradiance and laser thresholds. I aim to continue these efforts to construct the first phase transition model for Smith-Purcell superradiance, ultimately deriving order parameters and critical points consistent with experimental observations of superradiance, as in Fig. 2.

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