Path Integral for Seascape Noise*

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The effects of noise and non-uniformity on dynamics of populations are relevant and timely subjects of investigation. One form of variation is the time dependence of the reproduction rate (fitness), referred to as "seascape" noise; another is time-independent intrinsic dependencies of fitness on location (in the parlance of statistical physics, corresponding to annealed and quenched disorder, respectively). The former was studied recently and demonstrated to lead to novel universality classes for extinction and growth. In this paper, we show that there is a one-to-one correspondence between Seascape noise and a field theory and derive its action. This correspondence implies that the tools studied in field theories will carry over.

I. INTRODUCTION

Population dynamics has garnered diverse interest over the past few decades, with appearances in fields such as cancer research [1], forestry [2], pandemic modeling [3], wealth modeling [4], and many others. The multitude of data sets have been fitted to a number of empirical models [5] which call for good justification. Indeed, despite a rich literature on mathematical models for population growth [6–12], this field remains a fertile ground for theoretical explanations of the origins of various empirical models [2]. The simplest model of population growth is the logistic equation:

$$\frac{\mathrm{d}y}{\mathrm{d}t} = \mu y - ay^2 \ , \tag{1}$$

describing a population of size $y(t)$ that initially grows exponentially at rate μ (fitness), until resource limitation leads to saturation at $y(t \to \infty) = \mu/a$. A realistic population also has spatial dependence, and one of the simplest models that include spatial dependence is the *Fisher equation*, written as

$$\frac{\mathrm{d}y(x,t)}{\mathrm{d}t} = \mu y - ay^2 + D\nabla^2 y \ , \tag{2}$$

where D sets the rate for local migration.

While most empirical models focus on the behavior of an average population, there is underlying stochasticity that must be accounted for. One source of variation is due to the intrinsic randomness in reproduction events (whether a member of the population has zero, one, or several off-springs); this gives rise to the so-called *demographic noise* [13–15] and leads to interesting phenomena such as noise-stabilization [16], reversing the effect of deterministic selection [17], robust pattern formation [18], and rare extinction events [19, 20]. Mathematically, the magnitude of demographic noise is proportional to the square root of the population size, and it is thus most influential in small populations.

The focus of this work, however, is on extrinsic (e.g. environmental) factors that cause the reproduction rate itself to vary from location to location, and from time to time. Extending analogy from a random landscape, this type of *time-dependent* variation is referred to as *seascape noise*. Due to variations of the fitness term, this type of noise is mathematically expressed as a term proportional to the local population size $y(x,t)$ [21–23], leading to a stochastic differential equation of the form

$$\mathrm{d}y = (\mu y - ay^2 + D\nabla^2 y)\,\mathrm{d}t + \sigma y\,\mathrm{d}W \ , \tag{3}$$

with σ^2 being the variance of the seascape noise, and $\mathrm{d}W = \mathrm{d}W(x,t)$ is an uncorrelated Wiener process.

The presence of seascape noise can significantly alter the behavior of the population: ref. [21, 23] considered both a "mean-field" setting with the number of sites $N \to \infty$, while keeping spatial dependence by separating the sites into subclasses with identical properties. In the absence of seascape noise, it is easy to check that the steady-state mean behaves in the same way as the logistic equation, i.e. $\overline{y} = \sum_{i=1}^{N} y_i/N = \mu/a$. The average population vanishes linearly on approach to the extinction threshold for $\mu \to 0$. If there is no demographic noise, large seascape noise changes the extinction critical behavior from $\overline{y} \propto \mu^1$ to $\overline{y} \propto \mu^\beta$, with a critical exponent $\beta = 2D/\sigma^2$ (for $\sigma^2 > 2D$). Furthermore, in this regime, the probability distribution of the population broadens to the extent that the variance diverges, a feature not present with demographic noise. Spatial dependence has an important role: extinction is characterized by a critical exponent which depends on the characteristics of the subpopulation with the largest noise-to-migration ratio, regardless of the relative size of the subpopulation.

There is in principle no reason for the saturation term to take the quadratic form $S(y) = -ay^2$. Indeed, other forms have appeared in the literature; notably the Richards growth equation with $S(y) \propto y^\gamma$, with a fractional exponent γ fitted to the data [11, 12]. Originally introduced in describing plant growth [24–26], the

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Richards equation

$$\frac{\mathrm{d}\overline{y}}{\mathrm{d}t} = \mu\overline{y} - a\overline{y}^{\,\gamma} \;, \tag{4}$$

has been used in diverse contexts including modeling of pandemics [3, 27–30], bacterial growth [31], marketing [32], fisheries [33–35], forest growth [36–39], and agriculture [40, 41]. However, in the context of critical phenomena, introducing a *non-analytic* term at the outset is not legitimate. It is appropriate to extend the growth law by the inclusion of additional analytic terms in the expansion $S(y) = -ay^2 + a_3y^3 + a_4y^4 + \cdots$, but a non-analytic term $\propto y^\gamma$ requires justification. In Ref. [22], it was shown that eq. (4) emerges naturally upon averaging over migrating populations subject to seascape noise, as in eq. (3) for any such analytic $S(y)$.

While the aforementioned results [21–23] provide an intriguing connection, they consider the mean-field limit where the population size (and the number of "sites") tends to infinity. Although this assumption is sometimes valid, it does not hold for realistic systems, which are typically of finite size. Consequently, the application of the central limit theorem becomes unreliable, necessitating the adoption of stochastic differential equations to analyze such finite systems. However, when the number of sites falls within the intermediate regime, even numerical investigations can become intractable. In this journal paper, we propose an alternative approach that maps the dynamics of Seascape stochastic differential equations onto a "quantum" field theory, enabling the calculation of observables through perturbative analysis of the moments generating functionals. This framework draws upon various techniques from quantum field theory that are expected to be applicable in this context.

To gain intuition for the perturbative calculations, we first look into the case where we treat the entire population as one site, disregarding any spatial dependence. This is done in section II. Then, we consider how to generalize this to the case where population varies continuously over some space in section III.

II. ONE DIMENSIONAL SDE

When dealing with stochastic differential equations (SDEs), the objects of interest are the weight of a particular path $P[x(t)]$ and their corresponding moments. These are all captured by the moment generating functional Z. The goal of this section is to find Z for a one-dimensional seascape model using path integral. Seascape noise in continuum limit is an infinite dimesional generalization of this, discussed in the following section.

II.1. Realizing Path Integral

Consider a general Ito stochastic differential equation (SDE) in one dimension of the form:

$$\mathrm{d}x = f(x_t,t)\mathrm{d}t + g(x_t,t)\mathrm{d}W_t \;, \tag{5}$$

where W_t is a Weiner process (i.e Gaussian white noise), and x_t is the value of x at time t.

The goal of this section is to find the probability density functional (PDF) and moment generating functional for the stochastic variable $x(t)$. The initial condition can be set by

$$\mathrm{d}x = f(x_t,t)\mathrm{d}t + g(x_t,t)\mathrm{d}W_t + y\mathbb{1}_{t_0}(t) \;, \tag{6}$$

where $\mathbb{1}_{t_0}(t) = 1$ if $t = t_0$. Based on the intuition from path integral, this equation can be discretized into N time steps, $t_j = jh + t_0$:

$$x_{j+1} - x_j = f_jh + g_jw_j\sqrt{h} + y\delta_{j,0} \;, \tag{7}$$

Here, $x_0 = 0$ and w_j is a normally distributed discrete random variable with $\langle w_j\rangle = 0$ and $\langle w_jw_k\rangle = \delta_{j,k}$. Then, we can write the probability of a path conditioned on the initial condition y as

$$P[x|w;y] = \prod_{j=0}^{N} \delta(x_{j+1} - x_j - f_jh - g_jw_j\sqrt{h} - y\delta_{j,0}) \;. \tag{8}$$

In the more general case with a distribution of initial conditions, $P[y]$, the probability of a path is

$$P[x|w] = \int \mathcal{D}yP[x|w;y]P[y] \;. \tag{9}$$

Inserting the Fourier representation of the delta function and the PDF of w_i, we can write the probability of a path as

$$\begin{aligned} P[x|y] &= \int P[x|w;y]\prod_{j=0}^{N} P(w_j)\mathrm{d}w_j \\ &= \int \prod_{j=0}^{N} \frac{\mathrm{d}k_j}{2\pi} e^{-i\sum_j k_j(x_{j+1}-x_j-f_jh-y\delta_{j,0})} \\ &\quad \times \prod_{j=0}^{N} \frac{\mathrm{d}w_j}{\sqrt{2\pi}} e^{ik_jg_jw_j\sqrt{h}} e^{-\frac{1}{2}w_j^2} \;. \end{aligned} \tag{10}$$

After integrating out w_j, and changing to path integral formalism when $h \to 0$ and $N \to \infty$, we have

$$Z[J,\tilde{J}] = \int \mathcal{D}\tilde{x}(t)\mathcal{D}x(t)\, e^{-S[x,\tilde{x}] + \int \mathrm{d}t\, \tilde{J}(t)x(t) + \int \mathrm{d}t\, J(t)\tilde{x}(t)} \;, \tag{11}$$

with $ik_j = \tilde{x}(t)$. Then, the action takes the form

$$S[x,\tilde{x}] = \int \mathrm{d}t \left[\tilde{x}(t)\Big(\dot{x}(t) - f(x(t),t) - y\delta(t-t_0)\Big) - \frac{1}{2}\tilde{x}^2(t)g^2(x(t),t)\right] , \tag{12}$$

II.2. Application to Seascape Noise

Seascape noise in one dimension is a non-linear SDE:

$$\dot{x} = ax - bx^2 + y\delta(t-t_0) + \sigma x\eta(t) , \tag{13}$$

where a and b are constants, y is the initial population at time t_0, $\eta(t)$ is a Gaussian white noise process, and σ is the noise intensity. The noise is applied at time t_0 and is assumed to be zero for $t < t_0$. The noise is assumed to be small, so that the system is close to the deterministic system for $t < t_0$. The deterministic system has a stable fixed point at $x = a/b$.

The full action has the form:

$$S[x,\tilde{x}] = S_F[x,\tilde{x}] - y\tilde{x}(t_0) - b\int \mathrm{d}t\, \tilde{x}(t)x^2(t) - \frac{\eta^2}{2}\int \mathrm{d}t\, \tilde{x}^2 x^2 . \tag{14}$$

The linear part corresponding to $S_F[x,\tilde{x}] = \int \mathrm{d}t\, \tilde{x}(\dot{x} - ax)$ can be solved exactly. Then, the idea is to perform a perturbation expansion around this action. First, we find the moment generating functional:

$$Z[J,\tilde{J}] = \int \mathcal{D}x\, \mathcal{D}\tilde{x}\, e^{-S_F[x,\tilde{x}]} \times e^{\int \mathrm{d}t\, (-\tilde{x}bx^2 + \tilde{x}y\delta(t-t_0) + \tilde{x}^2x^2\sigma^2/2 + \tilde{J}x + x\tilde{J})} . \tag{15}$$

After Taylor expanding the exponential and completing Gaussian integrals using Wick's theorem, I obtain:

$$\begin{aligned} Z[J,\tilde{J}] =& Z_F[0,0]\bigg(1 + y\int \mathrm{d}\,t_1 G(t_1,t_0)\tilde{J}(t_1) \\ &+ \int \mathrm{d}t_1\mathrm{d}t_2\, \tilde{J}(t_1)J(t_2)G(t_1,t_2) \\ &- bD\int \mathrm{d}t_1\mathrm{d}t_2\mathrm{d}t_3\, G(t_2,t_1)G(t_3,t_3)\tilde{J}(t_2)\tilde{J}(t_3) \\ &- by^2\int \mathrm{d}t_1\mathrm{d}t_2\, G(t_1,t_0)^2G(t_2,t_1)\tilde{J}(t_2) \\ &+ y^2\int \mathrm{d}t_2\mathrm{d}t_1\, G(t_1,t_0)\tilde{J}(t_1)G(t_2,t_0)\tilde{J}(t_2) \\ &- 2bDy\int \mathrm{d}t_1\mathrm{d}t_2\mathrm{d}t_3\mathrm{d}t_4\, G(t_1,t_2)G(t_1,t_0) \\ &\times G(t_3,t_1)G(t_4,t_2)\tilde{J}(t_3)\tilde{J}(t_4) + \cdots\bigg) , \end{aligned} \tag{16}$$

where the propagator is $G(t,t') = H(t-t')e^{a(t-t')}$, with $H(t)$ being the Heaviside function. From this, one can calculate different moments as desired:

$$\begin{aligned} \langle x(t)\rangle =& yG(t,t_0) - bD\int \mathrm{d}t_1\mathrm{d}t_2\, G(t,t_1)G(t_1,t_2) \\ &- by^2\int \mathrm{d}t_1\, G(t,t_1)G(t_1,t_0)^2 + \cdots . \end{aligned} \tag{17}$$

As a consistency check, consider $t = t_0 + \Delta t$ for small Δt. The noise term has effect proportional to Δt^2 and can be ignored. The other two terms simplify to

$$\begin{aligned} \langle x(t)\rangle &= ye^{a(t-t_0)} - by^2\frac{e^{a(t-t_0)(e^{a(t-t_0)}-1)}}{a} \\ &= y + (ay - by^2)\Delta t , \end{aligned} \tag{18}$$

as expected.

The second cumulant of the population is

$$\begin{aligned} \langle x(s)x(t)\rangle_C =& D\int \mathrm{d}t_1\, G(s,t_1)G(t,t_1) \\ &- 2bDy\int \mathrm{d}t_1\mathrm{d}t_2\, G(t_1,t_2)G(t_1,t_0)G(s,t_1)G(t,t_2) \\ &- 2bDy\int \mathrm{d}t_1\mathrm{d}t_2\, G(t_1,t_2)G(t_1,t_0)G(t,t_1)G(s,t_2) \end{aligned} \tag{19}$$

III. INFINITE DIMENSIONAL SDE

In the field theory of population dynamics, each coordinate x corresponds to a population density $y(x,t)$, resulting in an infinite number of "sites". This parallels the concept of quantum field theory, which extends quantum mechanics to include an infinite number of degrees of freedom. In Seascape, there are much fewer symmetries, and instead, we possess knowledge of the equation of motion. Thus, rather than starting with the Landau-Ginzburg Hamiltonian, which yields the time-dependent Landau-Ginzburg equation of motion, we adopt an action-based approach similar to the one-dimensional case.

The infinite dimensional generalization of the aforementioned discussion is represented by the following action:

$$\begin{aligned} S[y,\tilde{y}] = \int \mathrm{d}^4x \bigg[&\tilde{y}(x)\Big(\dot{y}(x) - (ay(x) - by(x)^2 + D\nabla^2 y(x)) \\ &- y_0(x)\delta(t-t_0)\Big) - \frac{1}{2}\tilde{y}^2(t)y^2(x)\bigg] , \end{aligned} \tag{20}$$

where $x = (t,\vec{x})$ and $\mathrm{d}^4x = \mathrm{d}t, \mathrm{d}^3\vec{x}$. This action is a functional of the field $y(x)$ and its conjugate momentum $\tilde{y}(x)$. Notably, compared to the Landau-Ginzburg Hamiltonian's $\mathcal{P} \propto e^{-\beta\mathcal{H}}$, the key distinction lies in tracking field configurations in both space and time, rather than solely in space.

The comprehensive treatment of this field action will be addressed in future studies.

IV. CONCLUSION

In the above sections, we show that SDE can be mapped to path integrals and provide the corresponding actions for Seascape Fisher equations. THe interesting point about this mapping is that path integral can generalize to higher dimensions quite easily. It is also a systematic way to calculate an approximation for the solution to SDEs. Then, the various tools we studied in 8.334 can be applied.

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