Constraining the 3D Ising Model with the Conformal Bootstrap

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The Ising model is an enduring landmark in theoretical physics. In $d \leq 2$ it has been solved exactly, while in $d \geq 4$ mean-field techniques are effective; only the $d = 3$ case remains elusive. Though there have been many attempts using methods like ϵ-expansion, or high temperature expansion, it was not studied using the Conformal Bootstrap until [1] came out, which cast new possibility of solving the system in the continuum limit. In our project, we will start by reviewing the basics of CFTs in $d > 2$, including symmetry, OPEs and conformal blocks, and the bootstrap. Next, We review the approach and results in [1] to arrive at bounds for the weights of low-dimension operators with different spins. We then discuss the computational details that were used to produce these results.

I. INTRODUCTION

Ising model is an enduring landmark in theoretical physics. It was first motivated by ferromagnetism, but it was later learned that the critical behavior of many systems in the real world belong to the same universality class, with the most notably liquid-vapor transitions, and transitions in binary fluids and uniaxial magnets.

The Ising model was invented by Wilhelm Lenz. He gave it as a problem to his student Ernst Ising, who then solved the 1D case in his thesis and showed that there is no phase transition. The 2D case is much harder, and was not solved until a much later time when Onsager gave his exact solution. In dimensions greater than 4, the model can be very well described by mean-field theory. So the only unsolved mystery is the 3d case. It was proved by Istrail in 2000 in [2] that solving the 3D Ising model on the lattice is an NP-complete problem, and thus improbable. However, the possibility of finding a solution in the continuum limit is not ruled out.

Some typical approaches to this problem includes ϵ-expansion, high temperature expansion and Monte Carlo simulation, all of which have achieved decent agreement with the experiment. In 2012, David Simmons and his collaborators first brought the idea of studying the critical behavior of the 3d Ising model using conformal bootstraps. The conformal symmetry is a scale-invariant symmetry that only exists near the critical point and thus was not fully taken advantage of by the traditional approaches. It appeared that their results agree pretty well with the existing studies with even stronger constraining powers.

In this paper, we will first review what they did in their paper, and then see what else we can learn from higher spin operators. This paper is structured as follows. In section 2, we will breifly go over some basics of CFT and the idea of conformal bootstrap. In section 3, we will review what was done by DSD and his collaborators in their publications. And then in section 4, we discuss computational details to reproduce the previous work. Finally, we will conclude in section 5.

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II. CFT AND CONFORMAL BOOTSTRAP

While you can find many more detailed textbooks like [3], or lectures like [4] on CFT and Conformal Boostraps, we will review some of the very basics in this section that will come in handy in later discussions.

In general, CFT in d=2 behaves quite differently than d¿2 due to a larger symmetry group that is specific to d=2. Since we will be working in d=3 throughout this project, we will only discuss CFT in d¿2 in this section unambiguously.

A. Basics of CFT

A Conformal Field Theory(CFT) is a field theory equipped with a symmetry group called the conformal group, which is an extension to the typical Poincaré group that we might be more familiar with. The usual Poincaré group includes translations, rotations and Lorentz boosts. In addition to that, the conformal group also includes scalings and a so-called special conformal transformations(SCT), which can be thought of as inversions.

The conformal symmetry is often overlooked or not exploited in many early literature because it only emerges at the critical points. At the critical points, the correlation length diverges and, in addition to being Poincaré-invariant, the system obviously becomes scale invariant. Though it is still not fully understood why it should be conformal invariant at criticality in general, it seems to be the case, which will be assumed in the following discussion.

A continuum theory is often described in terms of local operators. In CFT, each operator is characterized by its scaling dimension Δ and a spin l, which can be further classified as either a primary, or descendent. The descendents can be constructed by taking derivatives of the primaries.

TABLE I. Some notable operators of 3d Ising model at criticality, taken from [1]

Operator	Spin l	Δ	Exponent
σ	0	0.5182(3)	Δ=1/2+η/2
ϵ	0	1.413(1)	Δ=3-1/ν
$T_{\mu\nu}$	2	3	n/a
$C_{\mu\nu\kappa\lambda}$	4	5.0208(12)	Δ=3+ω_{NR}

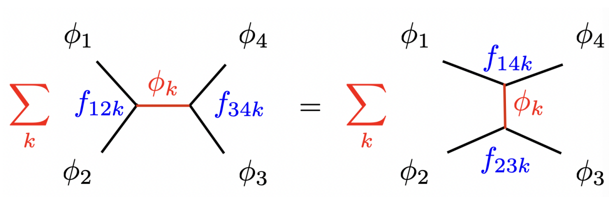

FIG. 1. The conformal bootstrap condition = associativity of the operator algebra. Taken from [1]

A few notable operators include σ and ϵ, which are operator versions of the Ising spin and the product of two neighboring spins on the lattice, the spin-2 stress energy tensor $T_{\mu\nu}$, the lowest dimension spin-4 operator $C_{\mu\nu\kappa\lambda}$ which has a small anomalous dimension, related to the critical exponent ω_{NR} measuring effects of rotational symmetry breaking on the cubic lattice. In I we listed some values characterizing the aforementioned operators, which are predicted and verified by various methods including ϵ-expansion, high temperature expansion and Monte Carlo simulations.

Another tool that we will use later is called the Operator Product Expansion(OPE). Any product of two operators in CFT can be decomposed into a linear combination of primaries and descendents. Since each descendent can be constructed using primaries, this relation can be conveniently written as

$$\phi_i(x_1)\phi_j(x_2) = \Sigma_k f_{ijk} C(x_1 - x_2, \partial_2)\phi_k(x_2) \qquad (1)$$

where the differential operator C are determined by the conformal invariance, and k only needs to sum over primaries by the above argument.

The coefficients f_{ijk} are called structure constants, or OPE coefficients. Together with the scaling dimension and spin, they comprise all the data needed to specify a CFT. In order to solve for a CFT, we need to solve for the CFT data, which leads to the idea of the Conformal Bootstrap.

B. The Conformal Bootstrap

The fundamental idea of the conformal bootstrap is shown schematically in Fig. 1. Consider a four point function of primaries

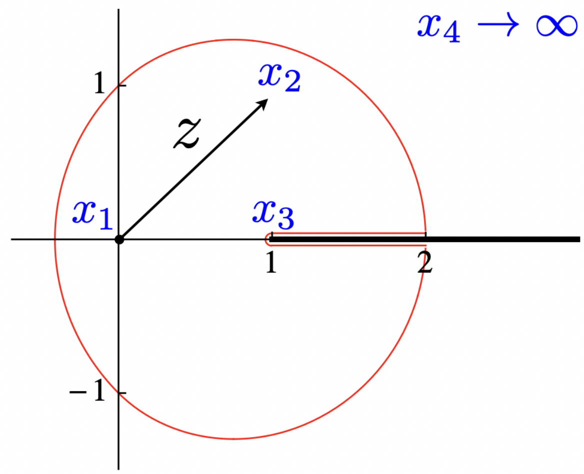

FIG. 2. We can fix the position of x_i using conformal freedoms. Then z is simply the position of x_2 in the complex plane. Taken from [1]

$$< \phi_1(x_1)\phi_2(x_2)\phi_3(x_3)\phi_4(x_4) > \qquad (2)$$

and if we expand it using OPE in the (12)(34) and (14)(23) channel to express it as a sum of two point correlators, we should expect the results to be the same, which leads to the relation

$$\Sigma_k f_{12k} f_{34k}(...) = \Sigma_k f_{14k} f_{23k}(...) \qquad (3)$$

For a four point function of four scalar operators ϕ_i with dimension Δ_i, its form is fixed by conformal invariance to be

$$\begin{aligned} &< \phi_1(x_1)\phi_2(x_2)\phi_3(x_3)\phi_4(x_4) > \\ &= (\frac{x_{24}}{x_{14}})^{\Delta_{12}} (\frac{x_{14}}{x_{13}})^{\Delta_{34}} \frac{g(u,v)}{x_{12}^{(\Delta_1+\Delta_2)} x_{34}^{(\Delta_3+\Delta_4)}} \end{aligned} \qquad (4)$$

where $x_{ij} \equiv x_i - x_j$ and $\Delta_{ij} \equiv \Delta_i - \Delta_j$ and g(u, v) is a function of the conformally invariant cross-ratios

$$\begin{aligned} u &= \frac{x_{12}^2 x_{34}^2}{x_{13}^2 x_{24}^2} \\ v &= \frac{x_{14}^2 x_{23}^2}{x_{13}^2 x_{24}^2} \end{aligned} \qquad (5)$$

with a form fixed by Eq. (3)

$$g_{u,v} = \Sigma_{\mathcal{O}} f_{12\mathcal{O}} f_{34\mathcal{O}} G_{\Delta,l}(u,v) \qquad (6)$$

where the sum runs over the exchanged primaries $\mathcal{O}$ with dimension Δ and spin l, and $G_{\Delta,l}(u,v)$ are called conformal blocks which are the main difficulty and focus of this project.

In order to compute the conformal blocks, we will first apply a change of variables as

$$\begin{aligned} u &= z\bar{z} \\ v &= (1-z)(1-\bar{z}) \end{aligned} \tag{7}$$

where the meaning of z is shown in Fig. 2. Using translational symmetry, we can always place one of the points at the origin, like x_1. Then using rotational symmetry w can put x_3 on the real axis. And using scaling and inversion symmetry we can move x_3 to (1, 0, 0, ...) and x_4 to inf. Thus, now we are only left with one degree of freedom which is the position of x_2, which we define to be z.

In the Appendix A of [1], it was found that, along the line of $z = \bar{z}$, $G_{\Delta,l}(z)$ satisfies a recursion relation,

$$\begin{aligned} &(l+D-3)(2\Delta+2-D)G_{\Delta,l}(z) = (D-2)(\Delta+l-1)G_{\Delta,l-2}(z) + \frac{2-z}{2z}(2l+D-4)(\Delta-D+2)G_{\Delta+1,l-1}(z) \\ &- \frac{\Delta(2l+D-4)(\Delta+2-D)(\Delta+3-D)(\Delta-l-D+4)^2}{16(\Delta+1-\frac{D}{2})(\Delta-\frac{D}{2}+2)(l-\Delta+D-5)(l-\Delta+D-3)}G_{\Delta+2,l-2}(z) \end{aligned} \tag{8}$$

whose base cases are given by

$$\begin{aligned} G_{\Delta,0}(z) &= (\frac{z^2}{1-z})^{\frac{\Delta}{2}}\, {}_3F_2(\frac{\Delta}{2}, \frac{\Delta}{2}, \frac{\Delta}{2}-\alpha; \frac{\Delta+1}{2}, \Delta-\alpha; \frac{z^2}{4(z-1)}) \\ G_{\Delta,1}(z) &= \frac{2-z}{2z}(\frac{z^2}{1-z})^{\frac{\Delta+1}{2}}\, {}_3F_2(\frac{\Delta+1}{2}, \frac{\Delta+1}{2}, \frac{\Delta+1}{2}-\alpha; \frac{\Delta}{2}+1, \Delta-\alpha; \frac{z^2}{4(z-1)}) \end{aligned} \tag{9}$$

For later convenience, we will actually have to compute the derivatives of $G_{\Delta,l}(z,\bar{z})$. Along the line of $z = \bar{z}$, we can express all the blocks as a linear combination of $G_{\Delta,0}(z)$ and $G_{\Delta,1}(z)$ by the recursion relation. Thus we need to calculate derivatives of hypergeometric functions, which we can calculate using another recursion relation that relates higher derivatives of the hypergeometric function to the 1st and 2nd derivatives of it,

$$(x\hat{D}_{a1}\hat{D}_{a2}\hat{D}_{a3} - \hat{D}_0\hat{D}_{b1-1}\hat{D}_{b2-1})\, {}_3F_2(a1, a2, a3; b1, b2; x) = 0 \tag{10}$$

where $\hat{D}_c \equiv x\partial_x + c$.

In order to get the derivatives transverse to the line $z = \bar{z}$, we will consider another change of variables first,

$$\begin{aligned} z &= (a+\sqrt{b})/2 \\ \bar{z} &= (a-\sqrt{b})/2 \end{aligned} \tag{11}$$

Since the conformal blocks are symmetric with respect to the exchange of z and $\bar{z}$, then the power series expansion of it transverse to $z = \bar{z}$ will only have even powers of $(z-\bar{z})$, and hence even powers of b. Then if we define the $\partial_m^a \partial_n^b$ derivative of the conformal block at $z = \bar{z} = 1/2$, the point that we want to expand around, as $h_{m,n}$, then another recursion relation is found in the Appendix C of [1],

$$\begin{aligned} &2(D+2n-3)h_{m,n} = \\ &2m(D+2n-3)[-h_{m-1,n} + (m-1)h_{m-2,n} + (m-1)(m-2)h_{m-3,n}] \\ &- h_{m+2,n-1} + (D-m-4n+4)h_{m+1,n-1} \\ &+ [2C_{\Delta,l} + 2D(m+n-1) + m^2 + 8mn - 9m + 4n^2 - 6n + 2]h_{m,n-1} \\ &+ m[D(m-2n+1) + m^2 + 12mn - 15m + 12n^2 - 30n + 20]h_{m-1,n-1} \\ &+ (n-1)[h_{m+2,n-2} - (D-3m-4n+4)h_{m+1,n-2}] \end{aligned} \tag{12}$$

By repeated applying this relation, any $h_{m,n}$ can be reduced to $h_{0,0}$ and $h_{1,0}$, which are just the $G_{0,0}$ and

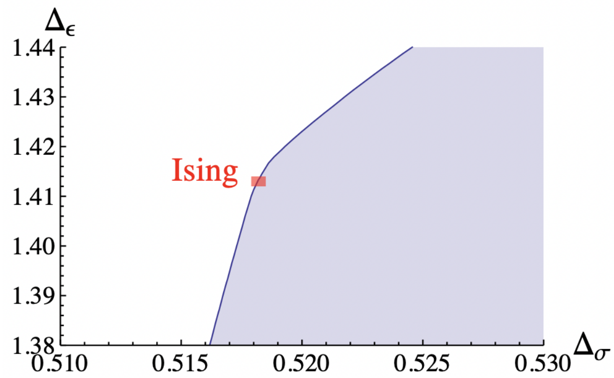

FIG. 3. The shaded area is the region allowed by crossing symmetry. Taken from [1]

$G_{1,0}$ that we found before. Thus it completely solved our problem of computing derivatives of the conformal blocks.

III. BOUNDS FOR THE 3D ISING MODEL

With all these tools already set up, we are now ready to look at the results reported in [1]. In order to get the bounds on the 3D Ising model, we will consider constraints from the four point function of the Ising spin operator $< \sigma\sigma\sigma\sigma >$. The conformal block expansion has the form

$$g(u,v) = \Sigma p_{\Delta,l} G_{\Delta,l}(u,v), p_{\Delta,l} \equiv f_{\Delta,l}^2 \geqq 0 \tag{13}$$

Since the operators are all identical, Eq. 3 now takes a simple form,

$$v^{\Delta_\sigma} g(u,v) = u^{\Delta_\sigma} g(v,u) \tag{14}$$

Plugging in Eq. 13, we get

$$\begin{aligned} u^{\Delta_\sigma} - v^{\Delta_\sigma} = \Sigma' p_{\Delta,l} [v^{\Delta_{sigma}} G_{\Delta,l}(u,v) \\ - u^{\Delta_{sigma}} G_{\Delta,l}(v,u)] \end{aligned} \tag{15}$$

where Σ' is the sum over all operators except the unit operator, which is put on the LHS. Then we can Taylor expand Eq. 15 around the point $z = \bar{z} = 1/2$ up to some fixed order. Then each Taylor coefficient will give you a linear equation, which forms a system of linear programming together with the positivity constraint on $p_{\Delta,l}$. Solving the system, we can find various bounds on the 3d Ising model.

A. Bounds on Δ_ϵ

First we want to see what is the maximal allowed value of Δ_ϵ. The result is shown in Fig. 3. The allowed region predicted by previous result as shown in I is plotted as the red rectangle.

We can conclude that the old methods do not contradict the Conformal invariance, though they are based on completely different principles, and using the new technique we can rigorously rule out a large amount of the parameter space. The 3d Ising model seems to lie closely to the boundary, if not on the boundary.

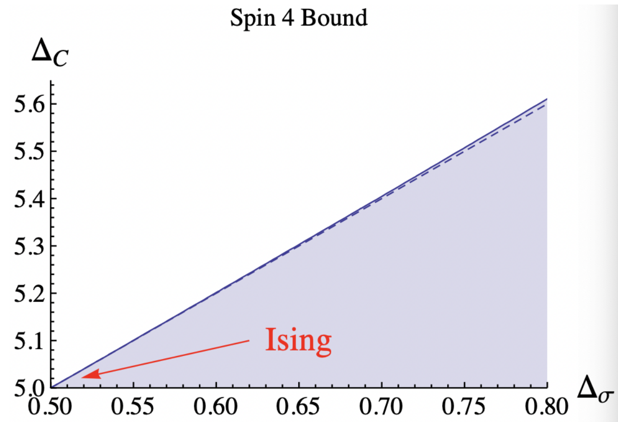

FIG. 4. The shaded area is the region allowed by crossing symmetry for the spin 4 operator. Taken from [1]

B. Bounds on Higher Spin Primaries

The first higher spin operator in the 3d Ising model is the spin 4 operator $C_{\mu\nu\kappa\lambda}$, which controls the leading effects of rotational symmetry breaking on a cubic lattice.

The bound is found numerically shown in Fig. 4. One notable result is that the Gaussian solution has $\Delta_C = 2\Delta_\sigma + 4$, which is respected and very close to the bound. The curve is fitted as

$$\Delta_C^{max} \simeq (2\Delta_\sigma + 4) + 0.1176(\Delta_\sigma - 1/2)^2 + O((\Delta_\sigma - 1/2)^3) \tag{16}$$

And this behavior can possibly be extended to higher spins. For example, for the lowest dimension spin 6 operator that controls the breaking of rotational symmetry on the tetrahedral lattice, the boundary closely follows the Gaussian line $\Delta_6 = 2\Delta_\sigma + 6$, with a fit

$$\Delta_6^{max} \simeq (2\Delta_\sigma + 6) + 0.1307(\Delta_\sigma - 1/2)^2 + O((\Delta_\sigma - 1/2)^3) \tag{17}$$

IV. COMPUTATIONAL METHOD

Now we turn to look at the computational method used to produce these results. In order to reproduce the results, we first need to code up the conformal blocks and

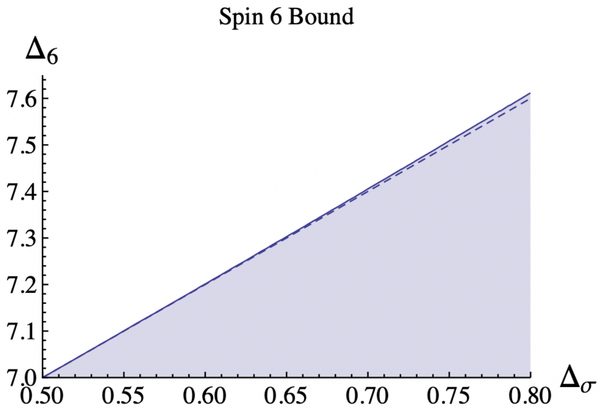

FIG. 5. The shaded area is the region allowed by crossing symmetry for the spin 6 operator. Taken from [1]

recursion relations introduced in Section 2. Though not as straightforward as it seems, it is just coding.

Then in order to place bounds on the 3d Ising model, we need to convert the crossing symmetry appreaed in Eq. 15 into a linear programming problem. We can write Eq. 15 as

$$0 = F_{0,0}^{\Delta_s}(u,v) + \Sigma' p_{\Delta,l} F_{\Delta.l}^{\Delta_\sigma}(u,v) \tag{18}$$

where $F_{\Delta,l}^{\Delta_s igma} \equiv v^{\Delta_\sigma} G_{\Delta,l}(u,v) - u^{\Delta_\sigma} G_{\Delta,l} v, u)$. Then if there exists a lieanr functional acting on functions of (u, v), such that

1. $\Lambda(F_{0,0}^{\Delta_\sigma}) = 1$ (normalization condition)

2. $\Lambda(F_{\Delta,l}^{\Delta_\sigma}) \geq 0$ for all Δ, l in the spectrum (positivity constraints).

Then if we act *Lambda* on Eq. 18, we get

$$0 = 1 + \Sigma' p_{\Delta,l}(positive\#) \tag{19}$$

Then there is no way that all $p_{\Delta,l}$ can stay positive and it is a sufficient condition to rule out this point. In practice, we consider Λ of the form

$$\Lambda : F(u,v) \Rightarrow \Sigma_{m+2n \leq 2n_{max}+1} \lambda_{m,n} \partial_a^m \partial_b^n F(a,b)|_{a=1,b=0} \tag{20}$$

where the variables are defined as before, $\lambda_{m,n}$ are real coefficients. Then we alraedy turned the problem into a linear programming problem. However, in order to obtain a sensible result, we need to increase n_{max}, which amounts to increase the total number of derivatives and computationaly heavy. So the result is hard to reproduce without better computing resource. But the idea is clearly outlined.

V. CONCLUSION

So, in conclusion, in this project, we reviewed a recent idea of applying the Conformal Bootstrap technique to the 3d Ising model from [1] in hope of placing bounds on the critical behavior of the model and potentially solving the model in the future. We can see that this technique does have strong constraining power even just by merely looking at the simplest four point function. However, the procedure is computationally heavy and might require better computational resources or algorithm for it to solve the model completely, if ever possible.

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