Melting of superfluid vortex crystal

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The topic of melting of a vortex lattice due to thermal fluctuations in 2D is well-studied and widely described in literature. However, if melting is caused by quantum fluctuations, the situation gets much more complicated and some of the aspects of this phenomena are still poorly understood. In this term paper I will make an overview of some results in a recent article by Dung Xuan Nguyen and Sergej Moroz, that suggests a criterion for such melting in rotating superfluid. After that I will derive how symmetries such as magnetic rotations and magnetic translations are realised in low-energy theory and how these results restrict possible nonlinear terms.

I. INTRODUCTION

Usually the melting of lattice occurs at finite temperature. For example, [1] suggests that melting in 2D is caused by defects called dislocations and disclinations. These defects are thermally excited and at high enough temperature the amount of these defects is enough to destroy the lattice. However, these defect can also emerge due to quantum fluctuations (QF). For atoms in most crystals, the lattice constant is very large and QF are not strong enough to cause melting. The vortex crystal in type-II superconductors is also a bad candidate, since inter-vortex distance would be small enough only at very high magnetic fields, for which superconducting phase is destroyed. One of the systems where such an effect is expected to be present is rotating superfluid. In this system angular velocity plays a role which is similar to the role of magnetic field in type-II superconductors, i.e. controlls the density of vortices. However, superfluid phase is more robust to rotation than superconducting phase to magnetic field, which will be discussed later in the text.

In recent article [2] (and a similar one, published at approximately the same time [3]), a mechanism of QF-induced vacancies and interstitials melting of a 2D vortex crystal is proposed. Besides many interesting results, such as a fracton-elasticity duality and relation of this problem to gravity, I will focus only on the calculation of a critical rotation frequency.

Finally, it is important to understand if the theory presented in [2] is consistent with the symmetries of the system, such as magnetic rotation and magnetic translation. Of cause, technically because of the presence of rotation axis these symmetries for the whole system are broken, but they are still present for a theory of weak deviations of vortex lattice from its equilibrium shape. Also, during the calculation I realised that the transformation rule that I found makes a constraint on possible nonlinearities in the theory.

II. QUANTUM MELTING OF VORTEX CRYSTAL

As I mentioned in the introduction, a rotating superfluid can be described as a stationary superfluid in uniform magnetic field. The paper [4] suggests that in lowest Landau level (LLL) approximation the action for such a theory can be written in the following way:

$$L = \frac{i}{2}\psi^\dagger D_t\psi - \frac{g^{ij}}{2m}D_i\psi^\dagger D_j\psi + \frac{gB}{4m}\psi^\dagger\psi + L_{int}. \tag{1}$$

Here $D_\mu = (\partial_\mu - iA_\mu)$, where A_μ is electromagnetic gauge field, $B = \epsilon_{ij}\partial_i A_j$ is magnetic field, g is a gyromagnetic factor. From this expression the authors defined the quantity a_μ which can be found from representation of supercurrent: $j^\mu = \frac{\delta S}{\delta A_m u} = \epsilon^{\mu\nu\rho}\partial_\nu a_\rho$. Using this gauge field a_μ the authors argued that the action for a vortex lattice formed in superfluid described by eqn (1) can be written as follows:

$$L = -\frac{Bn_0}{2}\epsilon_{ij}u^i\dot{u}_j + Be_iu^i - \frac{\lambda}{2}b^2 - E_{elastic}(u_{ij}). \tag{2}$$

In the above expression n_0 is an equilibrium superfluid density, u_i is a coarse-grained displacement field for vortex lattice, $e_i = \partial_t a_i - \partial_i a_0$ is an effective electric field, $b = \epsilon^{ij}\partial_i a_j$ is an effective magnetic field. The first term corresponds to a berry phase of vortices moving in superfluid, the second and the third ones represent the interaction of displacement field with the gauge field and the last term is simply an elastic energy of a triangular lattice $E_{elastic}(u_{ij}) = 2C_1(u_{kk})^2 + 2C_2(\tilde{u}_{ij})^2$, with $\tilde{u}_{ij} = u_{ij} - \frac{u_{kk}\delta_{ij}}{2}$ being a traceless part of $u_{ij} = (\partial_i u_j + \partial_j u_i)/2$.

A. Lifshits theory and Tkachenko modes

In [2] it was proposed that since for low-energy excitations there is an analog of Gauss law $\partial_i u^i = 0$, the deformation field can be written as a curl of a scalar field ϕ:

$$u^i = \epsilon^{ij}\frac{\partial_j\phi}{B}. \tag{3}$$

Substituting it in (2) and integrating out the gauge field, the authors showed that the effective Lagrangian for field ϕ is

$$L_{eff} = \frac{\dot{\phi}^2}{2\lambda} - \frac{C_2}{2B^2}(\epsilon^{jk}\partial_i\partial_k\phi + \epsilon^{ik}\partial_j\partial_k\phi)^2, \tag{4}$$

which reproduces the famous result for Tkachenko waves [5]:

$$\omega^2 = \frac{2C_2\lambda}{B^2}q^4. \tag{5}$$

Up to surface terms and normalising coefficients eqn (4) can be written in the following form:

$$L_0 = \frac{1}{2}\dot{\phi}^2 - \frac{\eta^2}{2}(\Delta\phi)^2. \quad (6)$$

Usually, when we write a Hamiltonian as a Taylor expansion of fields and its gradients, the lowest-order terms are Gaussian: $H = \frac{1}{2}\dot{\phi}^2 + \frac{c^2}{2}(\partial_i\phi)(\partial_i\phi)$. The case where the term quadratic in gradients is absent and rapidly oscillating field configurations are penalised by terms quartic in derivatives is called Lifshits theory. One could have guessed that ϕ is described by Lifshits theory simply because L_{eff} cannot depend on $\boldsymbol{u}$ due to translational invariance.

B. Condensation of vacancies

The key idea of [2] regarding lattice melting is the fact that defects such as vacancies and interstitials correspond to a vortex of field ϕ. Such vortices can be created by acting with an operator

$$\hat{O}_m(x) = \exp\Big(im\int d^2z\,\arg(z-x)\Pi(z)\Big), \quad (7)$$

where $\Pi(z)$ is a momentum operator, canonically conjugated to ϕ. Luckily, it is known how to calculate the spatial dimension for such operators, which is defined by relation

$$\langle\hat{O}_m(x)\hat{O}_m(x')\rangle \propto |x-x'|^{-2\Delta_m}. \quad (8)$$

According to rather technical calculation in [6], this scaling dimension is

$$\Delta_m = 2\pi\eta m^2. \quad (9)$$

When doing dynamic RG, one can see that if Δ_m is smaller than 2+2 (first 2 is a spatial dimension, which comes from integration of Lagrangian, 2 second is a dynamic critical exponent), then $\hat{O}_m(x)$ become relevant. This means that ground state of Lifshits theory is modified by any arbitrarily small perturbation containing $\hat{O}_m(x)$, i.e. vortex crystall is destroyed by defects.

Recalling the definition of η in terms of physical parameters and substituting it to criterion $\Delta_m < 4$, a more convenient condition can be written:

$$\frac{n_b}{n_v} \approx 8.2, \quad (10)$$

where n_b is a density of all bosons in the system and n_v is the density of vortexes. That means that vortex crystall melts when the number of vortexes reaches approximately 8 times smaller than the number of bosons. At the same time, the superfluidity itself is destroyed when $n_b \approx n_v$, so in that sense superfluid is more robust to rotation than superconductors are robust to magnetic fields.

III. COMMUTATION RELATIONS

Now let us find how magnetic rotations and translations act on field ϕ. First of all, we need commutation relations for generators of these symmetries. In [7] it was shown that in LLL approximation different components of coordinate and momentum operator become noncommuting:

$$[x_i, x_j] = i\theta\epsilon_{ij} \quad (11)$$
$$[p_i, p_j] = i\frac{1}{\theta}\epsilon_{ij},$$

where $\theta = -\frac{1}{B}$. Such an algebra emerges because now x and p operators are defined differently from usual quantum mechanics [8]. Namely, x is now a coordinate, projected on LLL and p is momentum of particle plus momentum of the field.

By considering the exact expression for translation operator, it can be shown (in [9]) that now momentum can be expressed in terms of coordinate:

$$p_i = \frac{1}{\theta}\epsilon_{ij}x_j. \quad (12)$$

Using this identity we can check that the commutation relation between x and p remain unchanged

$$[x_i, p_j] = \frac{1}{\theta}\epsilon_{jk}[x_i, x_k] = \frac{1}{\theta}\epsilon_{jk}i\theta\epsilon_{ik} = i\delta_{ij}. \quad (13)$$

Now let us define the angular momentum. Redefining eqn 14 of [7] in terms of new momentum variables we can see that z component of angular momentum can be written down as

$$J = \epsilon_{ij}x_ip_j + \frac{1}{2\theta}x_ix_i = -\frac{1}{2\theta}x_ix_i. \quad (14)$$

And finally, we need to compute the commutator of angular momentum and canonical momentum

$$[J, p_k] = -\frac{1}{2\theta^2}\epsilon_{kl}[x_ix_i, x_l] = i\epsilon_{kl}p_l. \quad (15)$$

IV. NONCOMMUTATIVE FIELD THEORY APPROACH TO SYMMETRIES

Once our theory became noncomutative, dealing with symmetry operators, which are exponents of generators can be problematic, since the naive Taylor series do not work. However, this problem can be solved if instead of ordinary operator multiplication we use Weil product and treat all the variables as commuting ones:

$$\hat{A}\cdot\hat{B} \to \hat{A}\star\hat{B} \equiv \hat{A}e^{\frac{i}{2}\theta\epsilon_{ij}\overleftarrow{\partial_i}\overrightarrow{\partial_j}}\hat{B}. \quad (16)$$

where $\overleftarrow{\partial_i}$ is a derivative that acts on the left term and $\overrightarrow{\partial_i}$ is the derivative acting on the right term. For more details see [8]. The overall algorithm for finding out how arbitrary operator $\hat{U}$ is transformed under the symmetry is the following:

- write down the symmetry generator using only coordinate operators
- exponentiate the generator treating all the coordinates as ordinary numbers to get symmetry operator
- write down the Weil product of symmetry operator and $\hat{U}$

A. Magnetic translation symmetry

Initially we defined field ϕ by claiming that its curl is a lattice displacement. It turns out that this transform is just a linearized version of $x_a \to e^{i\phi} \star x_a \star e^{-i\phi}$ (see [9]). Because of that all the symmetry operators (translations and rotations operators) will act on $e^{i\phi}$, but not on ϕ.

Let us apply the algorithm described in the previous section to translation symmetry. Magnetic translation by vector $\boldsymbol{c}$ acts on $e^{i\phi}$ in the following way:

$$e^{\phi} \to e^{-i\frac{1}{\theta}\epsilon_{ij}c_i x_j} \star e^{\phi}. \tag{17}$$

We can now expand the exponents and get a transform for field ϕ:

$$1 + i\phi + ... \to$$
$$\to (1 - i\frac{1}{\theta}\epsilon_{ij}c_i x_j + ...)(1 + \frac{i}{2}\theta\epsilon_{ij}\overleftarrow{\partial_i}\overrightarrow{\partial_j})(1 + i\phi + ...) \tag{18}$$

or

$$\phi \to \phi + \frac{1}{\theta}\epsilon_{ij}c_i x_j + \frac{1}{2}c_i\partial_i\phi + ... \tag{19}$$

The last term in equation above has a plus sign unlike in [9], which will be important later.

Now we can check the commutation relation for T_x and T_y. On one hand this commutator can be calculated directly (in leading order) using commutation relations for p_x and p_y:

$$[e^{ip_x a}, e^{ip_y b}]e^{i\phi} \approx -ab[p_x, p_y](1 + i\phi + ...) \approx -i\frac{ab}{\theta}. \tag{20}$$

On the other hand the commutator of translations acting on 1 gives zero, so the right hand side is a result of this commutator acting on $i\phi$. Hence we have:

$$[T_x(a), T_y(b)]\phi = -\frac{ab}{\theta}. \tag{21}$$

We can see that this is nothing else but a magnetic field flux through a rectangle with sights a and b. Based on that fact the authors of [9] deduced that **ϕ is equivalent to a phase of a condensate with removed vortex singularities**.

Now let us see if our transformation (19) is consistent with this commutator. By doing two consequent translations we can see that

$$T_x(a)T_y(b)\phi = \phi - \frac{1}{2\theta}ab + \frac{1}{\theta}(ay - bx) + \frac{1}{2}(a\partial_x\phi + b\partial_y\phi) + \frac{1}{4}ab\partial_x\partial_y\phi + ... \tag{22}$$
$$T_y(b)T_x(a)\phi = \phi + \frac{1}{2\theta}ab + \frac{1}{\theta}(ay - bx) + \frac{1}{2}(a\partial_x\phi + b\partial_y\phi) + \frac{1}{4}ab\partial_x\partial_y\phi + ... \tag{23}$$

which after subtraction reproduces (21). That means that equation (19) represents the action of magnetic translation group on ϕ and we do not need to consider nonlinear terms in that transformation.

B. Magnetic rotation symmetry

Magnetic rotation by angle α in xy plane is defined as an exponent of angular momentum operator $\hat{J}$:

$$R(\alpha) = e^{i\alpha J}. \tag{24}$$

using (14) we obtain the action of magnetic rotation on $e^{i\phi}$:

$$e^{i\phi} \to e^{i\alpha J} \star e^{i\phi}. \tag{25}$$

We want to keep linear terms in α, so we need to expand the first exponent to linear order. However, since J is quadratic in x, we now can leave first three terms in expansion of Weil product:

$$1 + i\phi - \frac{1}{2}\phi^2 + ... \to (1 - i\frac{\alpha}{2\theta}x_k x_k + ...)(1 + \frac{i}{2}\theta\epsilon_{ij}\overleftarrow{\partial_i}\overrightarrow{\partial_j} - \frac{\theta^2}{8}\epsilon_{ij}\epsilon_{kl}\overleftarrow{\partial_i}\overrightarrow{\partial_j}\overleftarrow{\partial_k}\overrightarrow{\partial_l} + ...)(1 + i\phi - \frac{1}{2}\phi^2 ...) \tag{26}$$

In the expression above we should be particularly careful, since one part of the terms in right hand side come from non-linear terms in transform of ϕ and another comes from transformed ϕ^2. In the end a new transformation rule for ϕ can be written as follows:

$$\phi \to \phi - \frac{\alpha}{2\theta}x_i x_i + \frac{\alpha}{2}\epsilon_{ij}x_i\partial_j\phi - \\ -\frac{\alpha\theta}{8}\partial_j\phi\partial_j\phi - \frac{\alpha\theta}{8}\phi\partial_j\partial_j\phi + ... \quad (27)$$

Similarly to the previous section, we can check if that transformation is consistent with commutation relations.

$$[R(\alpha), T(\boldsymbol{c})] \star e^{i\phi} \approx -\alpha c_i [J, p_i] \star (1 + i\phi) \approx \\ \approx -i\alpha c_i \epsilon_{ij} p_j (1 + \frac{i}{2}\theta\epsilon_{kl}\overleftarrow{\partial_k}\overrightarrow{\partial_l})(1 + i\phi) \quad (28)$$

substituting p_j from equation (12) and claiming that right hand side is the result of action of the commutator on $i\phi$ we see that

$$[R(\alpha), T(\boldsymbol{c})]\phi = \frac{\alpha}{\theta}c_i x_i - \frac{\alpha}{2}\epsilon_{kl}c_k\partial_l\phi + ... \quad (29)$$

which is a translation by vector $\alpha\epsilon_{ij}c_j$, as expected from commutator of J and p_k.

The calculation of commutator using eqns (19) and (27) is very laborious but straightforward and leads to the same results. I would like to empathise that without nonlinear terms in (27) magnetic translation and rotation algebra would not be realized.

V. ACTION OF THE SYMMETRIES ON LAGRANGIAN

Let us study how Lagrangian of Lifshits theory is modified under magnetic symmetries.In general case when we shift $\phi \to \phi + \phi_1$, the new Lagrangian of (6) becomes

$$L = L_0 + L_1 + L_2 \quad (30)$$

$$L_1 = \dot{\phi}\dot{\phi}_1 - \eta\Delta\phi\Delta\phi_1 \quad (31)$$

$$L_2 = \frac{1}{2}\dot{\phi_1}^2 - \frac{\eta^2}{2}(\Delta\phi_1)^2 \quad (32)$$

If we take ϕ_1 from equation (19):

$$\phi_1 = \frac{1}{\theta}\epsilon_{ij}c_i x_j + \frac{1}{2}c_i\partial_i\phi + ... \quad (33)$$

then the first term is obviously annihilated by derivatives, but the situation with second one is less trivial. During the calculation of action we integrate L_1 and we get

$$\int d^2x\Big(\dot{\phi}\dot{\phi}_1 - \eta\Delta\phi\Delta\phi_1\Big) = \\ = \frac{1}{2}c_i\int d^2x\Big(\dot{\phi}\partial_i\dot{\phi} - \eta\Delta\phi\partial_i\Delta\phi\Big) = \\ = \frac{1}{4}c_i\int d^2x\partial_i\Big(\dot{\phi}\dot{\phi} - \eta\Delta\phi\Delta\phi\Big) \quad (34)$$

which is an integral of total derivative and does not contribute to equations of motion.

L_2, however, is not a total derivative:

$$L_2 = \frac{1}{4}c_i c_j\big(\frac{1}{2}\partial_i\dot{\phi}\partial_j\dot{\phi} - \frac{\eta^2}{2}\partial_i(\Delta\phi)\partial_j(\Delta\phi)\big), \quad (35)$$

but it is still a higher order correction to (6).

If we now take ϕ_1 from (27):

$$\phi_1 = -\frac{\alpha}{2\theta}x_i x_i + \frac{\alpha}{2}\epsilon_{ij}x_i\partial_j\phi - \\ -\frac{\alpha\theta}{8}\partial_j\phi\partial_j\phi - \frac{\alpha\theta}{8}\phi\partial_j\partial_j\phi + ... \quad (36)$$

we can see that all the terms except for the first one generate higher-order (both in ϕ and ∂) corrections. The term $-\frac{\alpha}{2\theta}x_i x_i$ changes the Laplacian of ϕ:

$$\Delta\phi \to \Delta\phi - \frac{\alpha}{\theta} + ... \quad (37)$$

We can see that for such transformation (6) changes by a sum of constants and total derivatives and thus equations of motion are intact.

A. Nonlinear corrections to Lifshits theory

Equation (37) allows us to understand the structure of nonlinear terms in Lifshits theory. If the Lagrangian is an analytic function of $\dot{\phi}$ and $\Delta\phi$, then after magnetic rotation it transforms to

$$L(\dot{\phi}, \Delta\phi) \to L(\dot{\phi}, \Delta\phi - \frac{\alpha}{\theta}) = L(\dot{\phi}, -\frac{\alpha}{\theta}) + \frac{\partial L(\dot{\phi}, -\frac{\alpha}{\theta})}{\partial\Delta\phi}\Delta\phi + \frac{\partial^2 L(\dot{\phi}, -\frac{\alpha}{\theta})}{\partial(\Delta\phi)^2}\frac{(\Delta\phi)^2}{2} + ... \quad (38)$$

Since this theory should be invariant under magnetic rotations, we find that prefactor in front of $(\Delta\phi)^2$ is a constant independent of α. Although we derived (37) only for infinitesimal angles, we can see that during consequent rotation the constants add up, so this expression should work for any angle. Because of that the second derivative of L over $\Delta\phi$ is a fixed constant for all values of $\Delta\phi$ and thus all higher derivatives are zero. For example, this means that terms like $(\Delta\phi)^4$ are prohibited, because they generate corrections to $(\Delta\phi)^2$ term under magnetic rotations. This means that all nonlinearities can come only from terms that involve higher-order derivatives.

VI. CONCLUSION

In this term paper I made on overview of a theory that describes 2D vortex lattice melting due to quantum fluctuations. Surprisingly, this theory predicts that melting criterion is universal and depends only on $\frac{n_b}{n_v}$ ratio. Sadly, these results are hard to verify experimentally, because the critical rotation velocity is huge.

After that I found how magnetic rotation and translation act on field ϕ. I reproduced equation (19) which was obtained in [9] and corrected the sign of the last term, which is important for correct commutation relations. I did the same for magnetic rotations and by calculating commutators I identified the necessary amount of nonlinear terms that are needed for correct magnetic algebra. These results allowed us to understand which nonlinear terms in Lifshits theory are forbidden by symmetry.

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