Confinement and $\mathbb{Z}_2$ Gauge 3D Ising Model

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In this paper we provide an exposition to the problem of quark confinement in QCD. We introduce Wilson loops, which are a useful tool for calculating a potential between quarks and anti-quarks. In particular, we show that if a Wilson loop of a contour scales with the area of the contour in its exponent, we expect a linear "string-like" potential between quarks. Such behavior is in agreement with experimental observations of Regge trajectories that relate spins and masses of mesons. We show, on an example of a 3 dimensional $\mathbb{Z}_2$ Lattice Gauge Theory, how calculations of Wilson loops for in lattice field theories. In particular, we show that this model has unconfined and confined limits.

I. INTRODUCTION

Understanding quark confinement in Quantum Chromodynamics (QCD) is one of the most outstanding challenges in physics. One way to understand what we mean by quark confinement is that there are no isolated quark (asymptotic) particle states in QCD. This is in line with our experience and experimental observations, and we would like to understand this behavior from the theoretical point of view.

However, even formally defining confinement is troublesome. Part of the problem is that we don't even have formal proof that Non-Abelian gauge theories in 4 dimensions exist (this is a part of a Millennium Prize problem [5]). Another way to define it is to say that all the asymptotic particle states are singlets under the $SU(3)$ color gauge group of QCD (i.e. we never see color-charged particles). This is known as color confinement.

Alternatively, characterize quark confinement by saying that the potential between a static quark-antiquark pair grows linearly with the distance between them. Because of that, the pair will behave as a string, with infinite energy needed to break it. However, this is not true in QCD, as it is energetically favorable to create additional couple of quark and antiquark as we sufficiently increase the distance; this is known as a "string breaking" process. The potential in QCD will therefore be bounded from above because of the screening effect by quark matter fields. Nevertheless, linear potential characterization of confinement is meaningful as we take large quark masses (so that the pair creation is not energetically favorable). The benefit of this approach is that the potential can be related to the expectation value of an appropriate Wilson loop.

In Section 2 we are going to define Wilson loops and briefly introduce Lattice Field Theory. In Section 3, we will relate Wilson loops to the confining potential. This will give us the area law criterion for confinement. In Section 4, we show the experimental agreement of linear potential assumption for quarks at short distances. In section 5, we are going to study Wilson loops in a simplifying model of $\mathbb{Z}_2$ Lattice Gauge Theory in 3 dimensions. This model is dual to the 3D Ising model, so it will have a phase transition. This transition will correspond to a change in behavior of the Wilson loop, corresponding to the transition between confined and deconfined phase.

II. WILSON LOOPS [7]

By Elitzur's theorem, expectation value of any gauge non-invariant local observable must vanish. That means that we have to focus on gauge-invariant observables. For any given closed curve C in space time, and gauge field $A_\mu(x)$ in representation R of the gauge group, the Wilson loop is a gauge-invariant object defined as:

$$W_R(C) = \left\langle \mathrm{Tr}\left(\mathcal{P}\exp\left[i\oint_C A_\mu(x)dx^\mu\right]\right)\right\rangle, \tag{1}$$

where $\mathcal{P}$ is path-ordering.

For a lattice field theory, with lattice spacing a, we can define link variables $U_\mu(x)$ on the $(x, x+\hat{\mu})$ edge as:

$$U_\mu(x) = \exp\left(iaA_\mu(x)\right). \tag{2}$$

The Wilson loop is then the trace of the path-ordered product of link variables along the contour.

The Euclidean action for the gauge field in lattice theory is given by summing up Wilson loops of all plaquettes:

$$S_{gauge} = -\beta\sum_{x,\hat{\mu}<\hat{\nu}}\mathrm{Tr}\left(U_\mu(x)U_\nu(x+\hat{\mu})U^*_\mu(x+\hat{\nu})U^*_\nu(x)\right)+c.c. \tag{3}$$

Instead of adding quark fields (fermions), let us add complex scalar field in the fundamental representation, with matter action:

$$S_{\text{matter}} = -\sum_{\mathbf{x},\hat{\mu}}\left(\phi^\dagger(x)U_\mu(x)\phi(x+\hat{\mu}) + c.c.\right) + \sum_x (m^2+2D)\phi^\dagger(x)\phi(x), \tag{4}$$

where m^2 corresponds to the mass term, and everything else corresponds to a gauge invariant kinetic term.

III. AREA LAW FOR CONFINEMENT [3]

Let us take our contour to be a rectangle with sides R and T in an Euclidean Lattice field theory.

Let us define a gauge-invariant object:

$$Q_t = \phi^\dagger(0,t)\left(\prod_{i=0}^{R-1}U_\mu(i\hat{\mu},t)\right)\phi(R\hat{\mu},t), \tag{5}$$

which we interpret as a particle-antiparticle pair at distance R. So, if we want to calculate the amplitude that the pair is created and annihilated after Euclidean time T, we should calculate the expectation value $\langle Q_T^\dagger Q_0\rangle$. If we let $m^2 >> 1$ so that the mass term dominates, this expectation value turns out to be proportional to the Wilson loop of the rectangle (after integrating the scalar field)!

On the other hand,

$$\langle Q_T^\dagger Q_0\rangle = \frac{\langle 0|Q^\dagger e^{-HT} Q|0\rangle}{\langle 0|e^{-HT}|0\rangle} \tag{6}$$

$$= \frac{\sum_{n,m}\langle 0|Q^\dagger|n\rangle\langle n|e^{-HT}|m\rangle\langle m|Q|0\rangle}{\langle 0|e^{-HT}|0\rangle} \tag{7}$$

$$\propto e^{-\Delta E_{\min} T}, \tag{8}$$

as $T \to \infty$. Here, the relevant terms in the sum will be over the Hamiltonian eigenstates which contain the two particles at distance R, and the dominating term will be the term with the least energy. The energy of that state compared to the energy of the vacum will be the potential energy of the particles $V(R)$. Therefore,

$$W(R,T) \propto e^{-V(R)T}. \tag{9}$$

So if we have that for contours in general:

$$W(C) \propto e^{-\sigma Area(C)}, \tag{10}$$

that would imply that $V(R) \approx \sigma R$. In that case, we call σ the string tension.

IV. EXPERIMENTAL EVIDENCE OF CONFINEMENT [3]

One of the experimental evidence that the quark-anitquark potential is indeed linear are the Regge trajectories 1. These are experimental results which seem to suggest that for a given flavor of a meson, spin is proportional to the mass squared.

To see how linear potential answers this, we are going to approximate quark-antiquark pair as a stick of length $2R$ and linear density σ, spinning at relativistic speeds (so that its endpoints are going at the speed of light). This is string-like model, where potential grows linearly with the length. The mass of the spinning stick is:

$$m = 2\int_0^R \frac{\sigma dx}{\sqrt{1-v(x)^2}} = 2\int_0^R \frac{\sigma dx}{\sqrt{1-x^2/R^2}} = \pi\sigma R, \tag{11}$$

and the angular momentum is:

$$J = 2\int_0^R \frac{\sigma x v(x) dx}{\sqrt{1-v(x)^2}} = 2\frac{\sigma}{R}\int_0^R \frac{x^2 dx}{\sqrt{1-x^2/R^2}} = \frac{\pi}{2}\sigma R^2, \tag{12}$$

giving us relation:

$$J = \frac{1}{2\pi\sigma} m^2, \tag{13}$$

which is in agreement with the experimental observation.

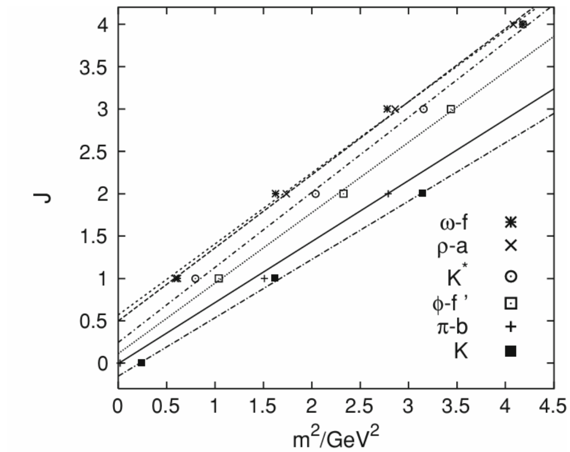

FIG. 1: Regge trajectories. Plot of spin J versus m^2 for mesons. Mesons of same flavor seem to lie on straight lines. The figure was taken from [8].

V. $\mathbb{Z}_2$ LATTICE GAUGE THEORY IN 3D [4]

As mentioned, studying confinement in QCD is hard. However, there are many simpler models in which confinement has been obtained explicitly. We are going to discuss a 3D model that was already mentioned in the lectures [4].

Let us focus on the simplest possible Gauge group: $\mathbb{Z}_2$. We can construct a lattice theory with $\mathbb{Z}_2$ gauge by setting link variables to be $\sigma = \pm 1$. The Hamiltonian is then:

$$\beta H = \kappa \sum_P \sigma_P^1 \sigma_P^2 \sigma_P^3 \sigma_P^4, \tag{14}$$

where the sum is over all plaquettes P, where in each plaquette we multiply the link variables. This Hamiltonian is invariant under the $\mathbb{Z}_2$ gauge transformation at each node which multiplies all the neighboring links by -1. Indeed, this local transformation always changes the even number of links in each plaquette, so it doesn't change the Hamiltonian.

Let us look at the behavior of Wilson loops for high and low κ. Notice that:

$$e^{\kappa\sigma^1\sigma^2\sigma^3\sigma^4} = \cosh(\kappa)(1+\tanh(\kappa)\sigma^1\sigma^2\sigma^3\sigma^4). \tag{15}$$

Therefore, for small κ, $W(C)$ will be dominated by the least number of plaquettes enclosing C (since each σ has to occur even number of times, we have to have a tiling of the curve C by the plaquettes). Therefore,

$$W(C) = \sum_{\partial A = C} (\tanh(\kappa))^{Area(A)} \to e^{\ln(\tanh(\kappa))Area(C)}, \tag{16}$$

where $Area(C)$ is the minimal area enclosing loop C. Therefore, we see the Area law behavior.

For large κ, the ground state is significantly preferred, with $\sigma \equiv 1$ an example of the ground state. Given that we can perform the gauge transformation at the each node, there are 2^N such ground states. Choose $\sigma \equiv 1$ for an example. Next leading terms are terms where we add one $\sigma = -1$ link. The leading contribution is then:

$$W(C) \approx \frac{2^N}{2^N} \frac{e^{3\kappa N}\left(1 + (3N - P_C)e^{-8\kappa} - P_C e^{-8\kappa}\right)}{e^{3\kappa N}\left(1 + 3Ne^{-8\kappa}\right)} \tag{17}$$

$$\approx \exp\left(-2P_C e^{-8\kappa}\right), \tag{18}$$

where P_C is the perimeter of the contour. Here we noticed that there are $(3N - P_C)$ links unrelated to the perimeter, for which each contribution in the expectation value expression will be $+e^{-8\kappa}$ compared to no excitation. Also, for each link on the contour we will have contribution of $-e^{-8\kappa}$. From this we see that the Wilson loop satisfies perimeter law scaling, implying that the potential is constant in large T limit, i.e. no confinement.

VI. DUALITY TO 3D ISING MODEL

In equation 16 we found an exact expression for a Wilson loop in $\mathbb{Z}_2$ 3D lattice gauge theory. Let us focus on the particular term in the partition function, for which we demand $C = 0$, i.e. only closed surfaces A contribute to the sum. This expression is similar to the low temperature expansion of the partition function for the 3D Ising model, where the contribution of an interface between pluses and minuses is e^{-2J} for each $+-$ link, giving $e^{-2J Area(A)}$ contribution in total. In this case, A is the surface of the dual lattice that goes between plus and minus domains. That means that we expect the dual relation between these two models to ne:

$$\ln(\tanh(\kappa)) = -2J \Leftrightarrow \sinh(2\kappa)\sinh(2J) = 1, \tag{19}$$

which is what we found in 2d Ising model duality as well.

VII. CONCLUSION

In this paper, we provided a brief introduction to the problem of quark confinement by studying Wilson loops. A particularly useful characterization of confinement is that Wilson loops scale with area in the exponent. This method gives us a tool to study confinement in lattice gauge theories. We showed what types of results we can expect from the simplest lattice gauge theory: with gauge group $\mathbb{Z}_2$ in 3 dimensions. In particular, we found that there is a confinement signature for high temperatures, while there is no confinement for low temperatures.

The study of quark confinement is an ongoing process. While exact results are hard, numerical tools in lattice gauge theory are increasingly useful in getting to understand QCD and confinement. Also, there are many manageable models (like compact QED of Polyakov [2]) where we have much better understanding of confinement. Study of this field keeps bearing fruit, as trying to understand confinement leads us to understanding Quantum Field Theories on a deeper level.

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