Even after all these simplifications, it is still not easy to calculate the Landau–Ginzburg partition function in Eq.(II.9). As a first step, we perform a *saddle point approximation* in which the integral in Eq.(II.9) is replaced by the maximum value of the integrand, corresponding to the *most probable* configuration of the field $\vec{m}(\mathbf{x})$. The natural tendency of interactions in a magnet is to keep the magnetizations vectors parallel, and hence we expect the parameter K in Eq.(II.8) to be positive[4] . Any variations in magnitude or direction of $\vec{m}(\mathbf{x})$ incur an 'energy penalty' from the term $K(\nabla\vec{m})^2$ in Eq.(II.8). Thus the field is uniform in its most probable configuration, and restricting the integration to this subspace gives

$$Z \approx Z_{sp} = e^{-\beta F_0} \int d\vec{m} \exp\left[-V\left(\frac{t}{2}m^2 + um^4 + \cdots - \vec{h}.\vec{m}\right)\right], \tag{II.10}$$

where V is the system volume. In the limit of $V \to \infty$ the integral is governed by the saddle point $\vec{m}$, which maximizes the exponent of the integrand. The corresponding saddle point free energy is

$$\beta F_{sp} = -\ln Z_{sp} \approx \beta F_0 + V \min\{\Psi(\vec{m})\}_{\vec{m}}, \tag{II.11}$$

where

$$\Psi(\vec{m}) \equiv \frac{t}{2}\vec{m}^2 + u\left(\vec{m}^2\right)^2 + \cdots - \vec{h}.\vec{m}. \tag{II.12}$$

The most likely magnetization will be aligned to the external field, i.e. $\vec{m}(\mathbf{x}) = \overline{m}\hat{h}$; its magnitude obtained from

$$\Psi'(\overline{m}) = t\overline{m} + 4u\overline{m}^3 + \cdots - h = 0. \tag{II.13}$$

Surprisingly, this simple equation captures the qualitative behavior at a phase transition.

While the function $\Psi(m)$ is analytic, and has no singularities, the saddle point free energy is Eq.(II.11) may well be non-analytic. This is because the minimization operation is not an analytic procedure, and introduces singularities as we shall shortly demonstrate. What justifies the saddle point evaluation of Eq.(II.10) is the thermodynamic limit of $V \to \infty$, and for finite V, the integral is perfectly analytic. In the vicinity of the critical point, the magnetization is small, and it is justified to keep only the lowest powers in the expansion of $\Psi(\vec{m})$. (We can later check self consistently that the terms left out of the expansion are indeed small corrections.) The behavior of $\Psi(m)$ depends strongly on the sign of the parameter t.

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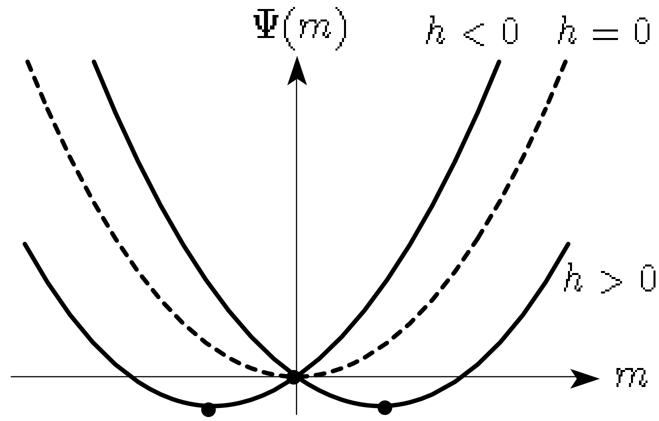
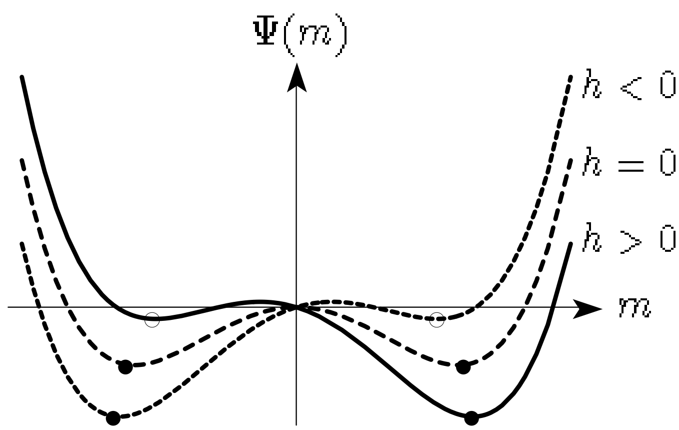
[4] This is also required by the stability condition.



II.1. The function $\Psi(m)$ for $t>0$, and three values of h. The most probable magnetization occurs at the minimum of this function (indicated by solid dots), and goes to zero continuously as $h \to 0$.

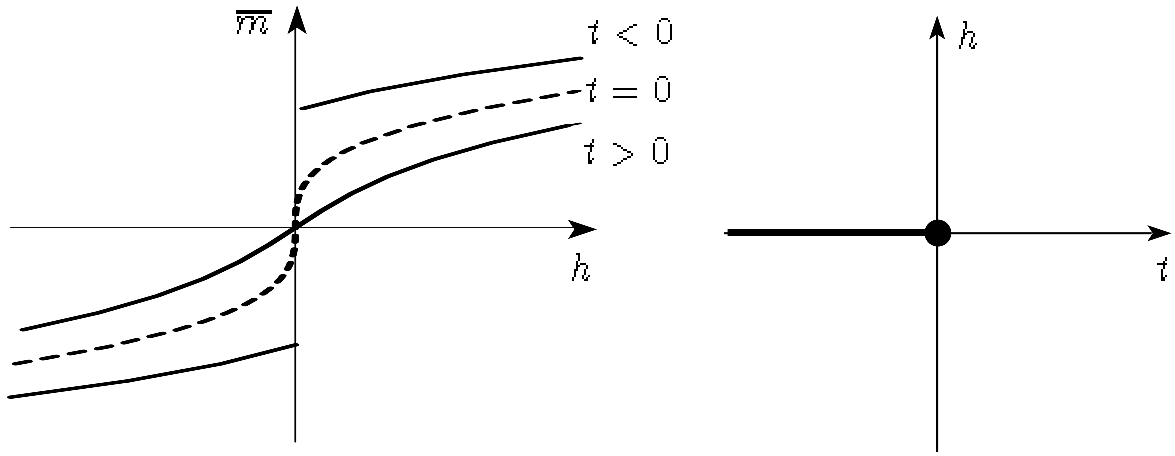
(1) For $t>0$, the quartic term is not necessary for stability and can be ignored. The solution to Eq.(II.13) is $\overline{m} = h/t$, and describes paramagnetic behavior. The most probable magnetization is aligned to the magnetic field, and vanishes continuously as $\vec{h} \to 0$. The susceptibility $\chi = 1/t$, diverges as $t \to 0$.

(2) For $t<0$, a quartic term with a positive value of u is required to insure stability (i.e. a finite magnetization). The function $\Psi(m)$ can now have two local minima, the global minimum being aligned with the field $\vec{h}$. As $\vec{h} \to 0$, the global minimum moves towards a non-zero value, indicating a spontaneous magnetization, even at $\vec{h} = 0$, as in a ferromagnet. The direction of $\vec{m}$ at $\vec{h} = 0$ is determined by the system's history, and can be realigned by an external field $\vec{h}$.



II.2. The function $\Psi(m)$ for $t<0$, and three values of h. The most probable magnetization occurs at the *global* minimum of this function (indicated by solid dots), and goes to zero continuously as $h \to 0$. The metastable minimum is indicated by open dots.

The resulting curves for the most probable magnetization $\overline{m}(h)$, are quite similar to those of Fig. The saddle point evaluation of the Landau–Ginzburg partition function thus results in paramagnetic behavior for $t > 0$, and ferromagnetic behavior for $t < 0$, with a line of phase transitions terminating at the point $t = h = 0$.



II.3. The magnetization curves obtained from the solution of Eq.(II.13) (left), and the corresponding phase diagram (right).

As noted earlier, the parameters $(t, u, K, \cdots)$ of the Landau Ginzburg Hamiltonian are analytic functions of temperature, and can be expanded around the critical point at $T = T_c$ in a Taylor series as

$$\left\{\begin{aligned} t(T, \cdots) =& a_0 + a_1(T - T_c) + \mathcal{O}(T - T_c)^2, \\ u(T, \cdots) =& u + u_1(T - T_c) + \mathcal{O}(T - T_c)^2, \\ K(T, \cdots) =& K + K_1(T - T_c) + \mathcal{O}(T - T_c)^2. \end{aligned}\right. \tag{II.14}$$

The expansion coefficients can be regarded as phenomenological parameters which can be determined by comparing to experiments. In particular, matching the phase diagrams in Figs. that t should be a monotonic function of temperature which vanishes at T_c, necessitating a_0 and $a_1 > 0$. Stability of the ferromagnetic phase in the vicinity of the critical point requires that u and K should be positive. The *minimal set of conditions* for matching the experimental phase diagram of a magnet to that obtained from the saddle point is

$$t = a(T - T_c) + \mathcal{O}(T - T_c)^2, \qquad \text{with} \quad (a, u, K) > 0. \tag{II.15}$$

It is of course possible to set additional terms in the expansion, e.g. a or u, to zero, and maintain the phase diagram and stability by appropriate choice of the higher order terms. However, such choices are not *generic*, and there is no reason for imposing more constraints than absolutely required by the experiment.