

Duality: Potts models & Percolation

1. *Clock model duality:* Consider spins $s_i = (1, 2, \dots, q)$ placed on the sites of a square lattice, interacting via the clock model Hamiltonian

$$\beta\mathcal{H}_C = - \sum_{\langle i,j \rangle} J((s_i - s_j)_{\text{mod}q}),$$

(a) Change from the N site variables to the $2N$ bond variables $b_{ij} = s_i - s_j$. Show that the difference in the number of variables can be accounted for by the constraint that around each plaquette (elementary square) the sum of the four bond variables must be zero modulus q .

(b) The constraints can be implemented by adding “delta-functions”

$$\delta[S_p]_{\text{mod}q} = \frac{1}{q} \sum_{n_p=1}^q \exp\left[\frac{2\pi i n_p S_p}{q}\right],$$

for each plaquette. Show that after summing over the bond variables, the partition function can be written in terms of the dual variables, as

$$Z = q^{-N} \sum_{\{n_p\}} \prod_{\langle p,p' \rangle} \lambda(n_p - n_{p'}) \equiv \sum_{\{n_p\}} \exp\left[\sum_{\langle p,p' \rangle} \tilde{J}(n_p - n_{p'})\right],$$

where $\lambda(k)$ is the discrete Fourier transform of $e^{J(n)}$.

(c) Calculate the dual interaction parameter of a Potts model, and hence locate the critical point $J_c(q)$.

(d) Construct the dual of the anisotropic Potts model, with

$$-\beta\mathcal{H} = \sum_{x,y} (J_x \delta_{s_{x,y}, s_{x+1,y}} + J_y \delta_{s_{x,y}, s_{x,y+1}});$$

i.e. with bonds of different strengths along the x and y directions. Find the line of self-dual interactions in the plane (J_x, J_y) .

2. Triangular/hexagonal lattice Ising model: For any *planar* network of bonds, one can define a geometrical dual by connecting the centers of neighboring plaquettes. Each bond of the dual lattice crosses a bond of the original lattice, allowing for a *local* mapping. The dual of a triangular lattice is a hexagonal (or honeycomb) lattice, and vice versa.

(a) Consider the Ising models on a hexagonal lattice with nearest neighbor interaction strength K_h . Note that the hexagonal lattice is *bipartite*, i.e. can be separated into two sublattices. In the partition function, do a partial sum over all spins in one sublattice. Show that the remaining spins form a triangular lattice with nearest neighbor interaction $K_t(K_h)$. (This is called the *star-triangle transformation*.)

(b) Show that the dual of a triangular Ising model is a hexagonal Ising model with the usual duality relation $\tilde{K}(K)$.

(c) By combining the previous results, obtain the critical couplings K_t^* and K_h^* of triangular and hexagonal lattices.

3. (Optional) Triangular/hexagonal lattice Potts model: The steps of the previous problem can be repeated for a general Potts model.

(a) Consider Potts spins ($s_i = 1, 2, \dots, q$) on a hexagonal lattice with nearest neighbor interaction $K_h \delta_{s_i, s_j}$. Perform the star-triangle decimation to show that the remaining spins form a triangular lattice with nearest neighbor interaction $K_t(K_h)$, *and* a three spin interaction $L(K_h)$. Why is L absent in the Ising model?

(b) What is the dual of the Potts model on the triangular lattice?

(c) Clearly, the model is not self-dual due to the additional interaction. Nonetheless, obtain the critical value such that $\tilde{K}_t(K_c) = K_c$. Then check that $L(K_c) = 0$, i.e. while the model in general is not self-dual, it is self-dual right at criticality, leading to the exact value of $K_c(q)$!

4. (Optional) Cubic lattice: The geometric concept of duality can be extended to general dimensions d . However, the dual of a geometric element of dimension D is an entity of dimension $d - D$. For example, the dual of a bond ($D = 1$) in $d = 3$ is a plaquette ($D = 2$), as demonstrated in this problem.

(a) Consider a clock model on a cubic lattice of N points. Change to the $3N$ bond variables $b_{ij} = s_i - s_j$. (Note that one must make a convention about the positive directions on the three axes.) Show that there are now $2N$ constraints associated with the plaquettes of this lattice.

(b) Implement the constraints through discrete delta-functions by associating an auxiliary variable n_p with each plaquette. It is useful to imagine n_p as defined on a bond of the dual lattice, perpendicular to the plaquette p .

(c) By summing over the bond variables in Z , obtain the dual Hamiltonian

$$\beta\tilde{\mathcal{H}} = \sum_p \tilde{J} (n_{12}^p - n_{23}^p + n_{34}^p - n_{41}^p),$$

where the sum is over all plaquettes p of the dual lattice, with $\{n_{ij}^p\}$ indicating the four bonds around plaquette p .

(d) Note that $\beta\tilde{\mathcal{H}}$ is left invariant if all the six bonds going out of any site are simultaneously increased by the same integer. Thus unlike the original model which only had a *global translation symmetry*, the dual model has a *local*, i.e. *gauge symmetry*.

(e) Consider a Potts gauge theory defined on the plaquettes of a four dimensional hyper-cubic lattice. Find its critical coupling $J_c(q)$.

Percolation

Fluids do not pass through a solid with a small concentration of holes. However, beyond a threshold concentration, the holes overlap, and the fluid can *percolate* through a connected channel in the material. *Percolation* is a classical *geometric* phase transition, and has been used as a model of many breakdown or failure processes. The loss of rigidity in an elastic network, conductivity in resistor nets, magnetization in diluted magnets are but a few examples.

In simple models of percolation, elements of a lattice (sites or bonds) are independently occupied with a probability p . A cluster is defined as a connected (by neighboring bonds) set of these occupied elements. At small p , only small clusters exist, and the probability that two sites, separated by a distance r , are connected to each other $U_p \equiv n_{12}^p - n_{23}^p + n_{34}^p - n_{41}^p p$, diverging at the *percolation threshold* p_c as $\xi(p) \sim |p_c - p|^{-\nu}$. An infinite cluster first appears at this threshold, and percolates through the (infinite) system for all $p > p_c$. The analog of the order parameter is the probability $P(p)$ that a site belongs to

this infinite cluster. On approaching p_c from above, it vanishes as $P(p) \sim |p_c - p|^\beta$. While the value of p_c depends on the details of the model, the exponents β and ν are *universal*, varying only with the spatial dimension d .

In the following problems we shall focus on *bond* percolation, i.e. p denotes the probability that a bond on the lattice is occupied. There is a mapping between percolation, and the $q \rightarrow 1$ limit of Potts models that is described in the review problems.

5. Duality has a very natural interpretation in percolation: If a bond is occupied, its dual is empty, and vice versa. Thus the occupation probability for dual bonds is $\tilde{p} = 1 - p \equiv q$. Since, by construction, the original and dual elements do not intersect, one or the other percolates through the system.

(a) The dual of a chain in which N bonds are connected in *series*, has N bonds connected in *parallel*. What is the corresponding (non-) percolation probability?

(b) The bond percolation problem on a square lattice is self-dual. What is its threshold p_c ?

(c) **(Optional)** Bond percolation in three dimensions is dual to plaquette percolation. Is it possible to have a percolating cluster that does not have solid integrity in $d = 3$?

6. Cayley trees: Consider a hierarchical lattice in which each site at one level is connected to z sites at the level below. Thus the n -th level of the tree has z^n sites.

(a) For $z = 2$, obtain a recursion relation for the probability $P_n(p)$ that the top site of a tree of n levels is connected to some site at the bottom level.

(b) Find the self-consistency condition governing the probability $P^*(p)$ that for infinitely many levels, the top layer is connected to some site at the bottom of the tree.

(c) Find the critical probability p_c , and the exponent β for the vanishing of $P^*(p)$ on approaching p_c .

(d) The one-dimensional chain corresponds to $z = 1$. Find the probability that end-points of an open chain of $N + 1$ sites are connected, and hence deduce the correlation length ξ .
