Beyond Spin Waves

1. *Nonlinear σ model with long–range interactions:* Consider unit n-component spins, $\vec{s}(\mathbf{x}) = (s_1, s_2, \cdots, s_n)$ with $|\vec{s}(\mathbf{x})|^2 = \sum_i s_i(\mathbf{x})^2 = 1$, interacting via a Hamiltonian

$$\beta\mathcal{H} = \int d^d\mathbf{x} \int d^d\mathbf{y}\, K(|\mathbf{x}-\mathbf{y}|)\, \vec{s}(\mathbf{x}) \cdot \vec{s}(\mathbf{y}) \quad .$$

(a) The long-range interaction, $K(x)$, is the Fourier transform of $Kq^\omega/2$ with $\omega < 2$. What kind of asymptotic decay of interactions at long distances is consistent with such decay? (Dimensional analysis is sufficient for the answer, and no explicit integrations are required.)

(b) Close to zero temperature we can set $\vec{s}\,(\mathbf{x}) = (\vec{\pi}\,(\mathbf{x}), \sigma(\mathbf{x}))$, where $\vec{\pi}\,(\mathbf{x})$ is an $n-1$ component vector representing *small fluctuations* around the ground state. Find the effective Hamiltonian for $\vec{\pi}\,(\mathbf{x})$ after integrating out $\{\sigma(\mathbf{x})\}$.

(c) Fourier transform the *quadratic part* of the above Hamiltonian focusing only on terms proportional to K, and hence calculate the expectation value $\langle \pi_i(\mathbf{q})\pi_j(\mathbf{q}')\rangle_0$.

We shall now construct a renormalization group by removing Fourier modes, $\vec{\pi}^{\,>}(\mathbf{q})$, with $\mathbf{q}$ in the shell $\Lambda/b < |\mathbf{q}| < \Lambda$.

(d) Calculate the coarse grained expectation value for $\langle\sigma\rangle_0^>$ to order of π^2 after removing these modes. Identify the scaling factor, $\vec{s}\,'(\mathbf{x}') = \vec{s}^{\,<}(\mathbf{x})/\zeta$, that restores $\vec{s}\,'$ to unit length.

(e) A simplifying feature of long–range interactions is that the coarse grained coupling constant is not modified by the perturbation, i.e. $\tilde{K} = K$ to all orders in a perturbative calculation. Use this information, along simple with dimensional analysis, to express the renormalized interaction, $K'(b)$, in terms of K, b, and ζ.

(f) Obtain the *differential* RG equation for $T = 1/K$ by considering $b = 1 + \delta\ell$.

(g) For $d = \omega + \epsilon$, compute T_c and the critical exponent ν to lowest order in ϵ.

Now consider the addition of the following two symmetry breaking terms

$$\beta\mathcal{H} \to \beta\mathcal{H} + \int d^d\mathbf{x}\left(h_1 s_1 + h_2 s_1^2\right) \, .$$

(h) Find the renormalization of the symmetry breaking field h_1 and identify the corresponding exponent y_h.

(i) Write the renormalization group equation for h_2 in the vicinity of the fixed point, and obtain the corresponding eigenvalue y_2.

(j) Find the divergence of the response function $\chi_2 \equiv \frac{d\langle s_1^2\rangle}{dh_2}$ as function of $t = T - T_c$ for $h_1 = h_2 = 0$.

(k) How does the correlation function $\langle s_1^2(\mathbf{x}) s_1^2(\mathbf{y})\rangle$ decay at the critical point?

(l) Show that symmetry breaking terms of the form $h_p s_1^p$ become irrelevant for $p > p^*$.

(m) If a short-range (e.g. nearest neighbor) interaction is added to the starting Hamiltonian does it change the critical behavior?

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2. *Symmetry breaking fields:* Let us investigate adding a term

$$-\beta\mathcal{H}_p = h_p \int d^2\mathbf{x} \cos\left(p\theta(\mathbf{x})\right),$$

to the XY model. There are a number of possible causes for such a symmetry breaking field: $p = 1$ is the usual 'magnetic field,' $p = 2$, 3, 4, and 6 could be due to couplings to an underlying lattice of rectangular, hexagonal, square, or triangular symmetry respectively. As $h_p \to \infty$, the spin becomes discrete, taking one of p possible values, and the model becomes equivalent to clock models.

(a) Assume that we are in the low temperature phase so that vortices are absent, i.e. the vortex fugacity y is zero (in the RG sense). In this case, we can ignore the angular nature of θ and replace it with a scalar field ϕ, leading to the partition function

$$Z = \int D\phi(\mathbf{x}) \exp\left\{-\int d^2\mathbf{x}\left[\frac{K}{2}(\nabla\phi)^2 + h_p\cos(p\phi)\right]\right\}.$$

This is known as the sine–Gordon model, and is equivalent to the roughening transition. Obtain the recursion relations for h_p and K.

(b) Show that once vortices are included, the recursion relations are

$$\begin{cases} \dfrac{dh_p}{d\ell} = \left(2 - \dfrac{p^2}{4\pi K}\right) h_p, \\ \dfrac{dK^{-1}}{d\ell} = -\dfrac{\pi p^2 h_p^2}{4} K^{-2} + 4\pi^3 y^2, \\ \dfrac{dy}{d\ell} = (2 - \pi K)\, y. \end{cases}$$

(c) Show that the above RG equations are only valid for $\frac{8\pi}{p^2} < K^{-1} < \frac{\pi}{2}$, and thus only apply for $p > 4$. Sketch possible phase diagrams for $p > 4$ and $p < 4$. In fact $p = 4$ is rather special as there is a marginal operator h_4, and the transition to the 4-fold phase (cubic anisotropy) has continuously varying critical exponents!

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3. *The XY model in $2+\epsilon$ dimensions:* The recursion relations of the XY model in two dimensions can be generalized to $d = 2+\epsilon$ dimensions, and take the form:

$$\begin{cases} \dfrac{dT}{d\ell} = -\epsilon T + 4\pi^3 y^2 \\ \dfrac{dy}{d\ell} = \left(2 - \dfrac{\pi}{T}\right) y \end{cases} .$$

(a) Calculate the position of the fixed point for the finite temperature phase transition.

(b) Obtain the eigenvalues at this fixed point to *lowest* contributing order in ϵ.

(c) Estimate the exponents ν and α for the superfluid transition in $d = 3$ from these results. [Be careful in keeping track of only the lowest nontrivial power of ϵ in your expressions.]

(d) The symmetry breaking term $h_p \int d^d\mathbf{x} \cos\left(p\theta(\mathbf{x})\right)$ follows the RG equation

$$\frac{dh_p}{d\ell} = \left(d - \frac{p^2}{4\pi} T\right) h_p .$$

Show that h_p is irrelevant for $p > p^*$.

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4. *Inverse-square interactions:* Consider a scalar field $s(x)$ in one-dimension, subject to an energy

$$-\beta\mathcal{H}_s = \frac{K}{2}\int dxdy \frac{s(x)s(y)}{|x-y|^2} + \int dx \Phi[s(x)].$$

The local energy $\Phi[s]$ strongly favors $s(x) = \pm 1$ (e.g. $\Phi[s] = g(s^2-1)^2$, with $g \gg 1$).

(a) With $K > 0$, the ground state is ferromagnetic. Estimate the energy cost of a single domain wall in a chain of length L. You may assume that the transition from $s = +1$ to $s = -1$ occurs over a short distance cutoff a.

(b) From the probability of the formation of a single kink, obtain a lower bound for the critical coupling K_c, separating ordered and disordered phases.

(c) Show that the energy of a dilute set of domain walls located at positions $\{x_i\}$ is given by

$$-\beta\mathcal{H}_Q = 4K\sum_{i<j} q_i q_j \ln\left(\frac{|x_i - x_j|}{a}\right) + \ln y_0 \sum_i |q_i|,$$

where $q_i = \pm 1$ depending on whether $s(x)$ increases or decreases at the domain wall. (Hints: Perform integrations by part, and coarse-grain to size a. The function $\Phi[s]$ only contributes to the core energy of the domain wall, which results in the fugacity y_0.)

(d) The logarithmic interaction between two opposite domain walls at a large distance L, is reduced due to screening by other domain walls in between. This interaction can be calculated perturbatively in y_0, and to lowest order is described by an effective coupling (see later)

$$K \to K_{eff} = K - 2Ky_0^2 \int_a^\infty drr \left(\frac{a}{r}\right)^{4K} + \mathcal{O}(y_0^4). \tag{1}$$

By changing the cutoff from a to $ba = (1 + \delta\ell)a$, construct differential recursion relations for the parameters K and y_0.

(e) Sketch the renormalization group flows as a function of $T = K^{-1}$ and y_0, and discuss the phases of the model.

(f) Derive the effective interaction given above as Eq.(1). (Hint: This is somewhat easier than the corresponding calculation for the two-dimensional Coulomb gas, as the charges along the chain must alternate.)

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5. (Optional) *Spatially anisotropic $\mathcal{O}(n)$ model:* n component unit vectors $\vec{s}_i$ on sites of a three dimensional simple cubic lattice interact with their nearest neighbors, with *anisotropic* interactions of strengths K_x, K_y, and K_z for neighbors along the x, y, and z directions, respectively. Consider the high temperature expansion, such that

$$\exp\left[K_s(\vec{s}_i \cdot \vec{s}_j)\right] \approx 1 + K_s \vec{s}_i \cdot \vec{s}_j + \mathcal{O}(K_s^2),$$

for all three directions (i.e. $s = x$, y, or z).

(a) The lowest order graphs for a high temperature expansion of the correlation function $\langle \vec{s}_{\mathbf{0}} \vec{s}_{\mathbf{r}} \rangle$ are paths connecting the origin $\mathbf{0} = (0, 0, 0)$ to the point $\mathbf{r} = (x, y, z)$. What is the contribution of a single such path of ℓ_x, ℓ_y, and ℓ_z steps along the x, y, and z directions, respectively?

(b) Treating the paths as phantom (Markovian), use Fourier transforms to find the eigenvalues of the matrix T^ℓ whose components give the weight of ℓ step random walks between any two points on the cubic lattice.

(c) Find the total contribution $W(x, y, z)$, *of paths of all lengths* from $\mathbf{0}$ to $\mathbf{r}$ as an appropriate Fourier transform.

(d) When does the summation over phantom paths cease to make sense (corresponding to the phase transition point in this approximation).

(e) What is the shape of curves, parametrized by $\mathbf{r} = (x, y, z)$, such that $\langle(\vec{s}_{\mathbf{0}} \cdot \vec{s}_{\mathbf{r}})\rangle$ is a constant (for large x, y and z, and close to the above transition point)?

(f) What is the critical exponent ν for the divergence of the correlation length in this phantom approximation?

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