8.581 - Information, Entropy and the Brain

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Problems for Lecture 1

1. Maximum entropy properties of Gaussians

In order to minimize the entropy

$$S(x) \equiv - \int dx P(x) \log P(x), \tag{1}$$

subject to $\int P dx = 1$, $\int x P dx = \langle x \rangle$, $\int x^2 P dx = \alpha^2$, note $\alpha^2 = \langle x \rangle^2 \equiv \mu^2$ by the $Cauchy - Schwartz$ inequality, and set $\sigma^2 = \alpha^2 - \mu^2$. Apply then the variational method to the *Lagrange multiplier constrained functional* H in curly brackets

$$\delta \int \{-P(x) \log P(x) - \lambda_1 P(x) - \lambda_2 P(x) x - \lambda_3 x^2 P(x)\} = 0 \tag{2}$$

$$\frac{\partial H}{\partial P} - \partial_x (\frac{\partial H}{\partial \dot{P}}) = 0 \rightarrow \log P(x) - 1 - \lambda_1 - x\lambda_2 - x^2 \lambda_3 = 0 \tag{3}$$

Thus P is gaussian.

$$P(x) = \exp(\lambda_1 + \lambda_2 x + \lambda_3 x^2) = \frac{1}{\sqrt(2\pi\sigma^2)} \exp(-\frac{(x - \langle x \rangle)^2}{2\sigma^2}) \tag{4}$$

where $\lambda_1 = -\log(2\pi\sigma^2) - \frac{\langle x \rangle^2}{2\sigma^2}$ $\lambda_2 = \frac{\langle x \rangle}{2\sigma^2}$ $\lambda_3 = -\frac{1}{2\sigma^2}$ are the *Lagrange multipliers* corresponding to the normalization, average and variance constrains.

2. Information with Gaussian signals and noise

Let s,η be Gaussian input the signal and noise. Their sum $y = s + \eta$ is at the output.

$$P_\nu(\nu) = \frac{1}{\sqrt(2\pi\sigma_\nu^2)} \exp(-\frac{(\nu - \langle \nu \rangle)^2}{2\sigma_\nu^2}) \tag{5}$$

where $\nu = s, \eta$, and y is the output signal:

$$P_y(y) = \frac{1}{\sqrt(2\pi(\sigma_s^2 + \sigma_\eta^2)} \exp(-\frac{(y - \langle s \rangle - \langle \eta \rangle)^2}{2(\sigma_s^2 + \sigma_\eta^2))}) \ . \tag{6}$$

The mutual Information exchanged is

$$I(y \rightarrow s) = H(y) - H(y|s) = H(y) - H(\eta|s) = H(y) - H(\eta) \tag{7}$$

$$I(y \to s) = \int dyds P(y|s) P_s(s) \log \Big[\frac{P(y|s)}{P_y(y)}\Big]$$

$$= \int dsd\eta P_\eta(y-s) P_s(s) \Big\{ \log P_\eta(y-s) - \log P_y(y) \Big\} \tag{8}$$

$$= \langle \log P_\eta(\eta) \rangle_\eta - \langle \log P_y(y) \rangle_y \tag{9}$$

$$= -\frac{1}{2} \log (2\pi\sigma_\eta^2) - \langle \frac{(\eta - \langle\eta\rangle)^2}{2\sigma_\eta^2} \rangle_\eta + \frac{1}{2} \log (2\pi\sigma_y^2)) + \langle \frac{(y - \langle y\rangle)^2}{2\sigma_y^2} \rangle_y. \tag{10}$$

Then

$$I(y \to s) = \frac{1}{2} \log (\frac{\sigma_y^2}{\sigma_\eta^2}) = \frac{1}{2} \log (1 + \frac{\sigma_s^2}{\sigma_\eta^2})$$

where $SNR = \frac{\sigma_s^2}{\sigma_\eta^2}$ is the *Signal to Noise* ratio.

3. Multicomponent signals

Consider now a multicomponent signal

$$y_i = s_i + \eta_i$$

$$\langle s_i s_j \rangle = S_{ij} \quad \langle \eta_i \eta_j \rangle = N_{ij} \tag{11}$$

So the expression (8) becomes, in general

$$I(\vec{y} \to \vec{s}) = \int d\vec{y} d\vec{s} P(\vec{y}|\vec{s}) P_{\vec{s}}(\vec{s}) \log \Big[\frac{P(\vec{y}|\vec{s})}{P_{\vec{y}}(\vec{y})}\Big] \tag{12}$$

$$= -\frac{1}{2} \log [detN] + \frac{1}{2} \log \det(N+S) - \frac{1}{2} \langle s^T S^{-1} s \rangle_s + \frac{1}{2} \langle y^T (S+N)^{-1} y \rangle = \tag{13}$$

$$\frac{1}{2} [\log det(N+S) - \log detN] = \frac{1}{2} \log det(1 + N^{-1}S) \tag{14}$$

Since, for a given matrix A:

$$\log detA = Tr \log A \tag{15}$$

then

$$I = \frac{1}{2} Tr \log(1 + N^{-1}S) \tag{16}$$

4. Time dependent formulation

Consider $y(t) = s(t) + \eta(t)$ and define $y_T(t)$, $\eta_T(t)$, $s_T(t)$ to be the same functions, restricted in the time interval $\{0, T\}$. The continuous-time Fourier series of these functions are, e.g for the noise and input signals N and S:

$$S_T(t) = S_0 + \sum_n^\infty S_T^n(t) \qquad N_T(t) = N_0 + \sum_n^\infty N_T^n(t)$$

and S_T^n, N_T^n the $\frac{2n\pi}{T} = \omega_n$ components of the signal. The frequencies $\frac{2\pi}{T}m$ and $\frac{2\pi}{T}n$ are independent so that S_T^n and S_T^m share no information. (the two processes N and S are called *Wiener* processes in this case). Also

$$\langle s_i s_j \rangle = \langle s(t)s(t') \rangle = C(t-t')$$

$$s(t) = \sum_{\omega = \pm\frac{n\pi}{T}} \tilde{s}(\omega) e^{i\omega t} \tag{17}$$

$$\langle s(t)s(t') \rangle = \sum_{\omega_1,\omega_2} \langle \tilde{s}(\omega_1)\tilde{s}(\omega_2) \rangle e^{i\omega_1 t} e^{i\omega_2 t'} \delta(t-t') \tag{18}$$

The mutual information is

$$I_T(y \to s) = \sum_{n=0}^{\infty} [H_T(y_T^n) - H_T(N_T^n)] \tag{19}$$

$$I = \int DsDyP(y(t),s(t)) \log\left[\frac{P(y(t)|s(t))}{P(y(t))P(s(t))}\right] \tag{20}$$

$$= \frac{1}{2} \sum_{\omega=\frac{2\pi n}{T}}^{\infty} \frac{d\omega}{2\pi}\frac{1}{2} \log(1 + \frac{\langle \tilde{s}(\omega) \rangle^2}{\langle \tilde{\eta}(\omega) \rangle^2}) \tag{21}$$

Hence

$$R_{info} \equiv \lim_{T\to\infty} \frac{I(y \to s)}{T} = \frac{1}{2} \int \frac{d\omega}{2\pi} \log\,(1 + \frac{S(\omega)}{N(\omega)}). \tag{22}$$

5. Lagrange Multipliers

To maximize R_{info} , subject to the constrain $\int \frac{d\omega}{2\pi} S(\omega) = \langle s^2 \rangle$, write

$$\delta \int \frac{d\omega}{2\pi}\{ \log\,(1 + \frac{S(\omega)}{N(\omega)}) - \lambda S(\omega)\} = 0 \tag{23}$$

$$\int \frac{d\omega}{2\pi}\{\frac{1}{S(\omega)+N(\omega)} - \lambda\}\delta S = 0 \tag{24}$$

$$S(\omega) + N(\omega) = \frac{1}{\lambda} = const. \tag{25}$$

The condition is not satisfied for all values of ω. The maximal frequency range for $S(\omega)$ is obtained filling the dashed region in fig.1, given the constrain of the total dashed area to be $\langle s^2 \rangle$.

Problems for lecture 2

1. Steady state solution and the linear response regime.

Consider the two, coupled, differential equations

$$C\frac{dV}{dt} = -gn(V - V_{ion}) - G_{leak}(V - V_{rest}) + I \tag{26}$$

$$\frac{dn}{dt} = -\frac{1}{\tau(V)}[n - n_{eq}(V)] \tag{27}$$

$$n = n_{eq}(V_0) \tag{28}$$

$$n_{eq}(V_0) = \frac{G_{leak}(V_0 - V_{rest})}{g(V_0 - V_{ion})} \tag{29}$$

also note that $V_{rest} > V_0 > V_{ion}$. Consider then small-signal variations of these variables, in response to an infinitesimal current I.

In the Linear response regime:

$$C(\dot{\delta V}) = -g(V - V_{ion})\delta n - gn_{eq}\delta V + G_{leak}\delta V + \delta I \tag{30}$$

$$(\dot{\delta n}) = -\frac{1}{\tau(V_0)}(\delta n - \frac{\partial n_{eq}}{\partial V}\Big|_{V_0}\delta V). \tag{31}$$

$$-\imath\omega C\delta\tilde{V} = -g(V_0 - V_{ion})\delta\tilde{n} - (gn_{eq} + G_{leak})\delta\tilde{V} + \delta\tilde{I} \tag{32}$$

$$-\imath\omega\delta\tilde{n} = -\frac{1}{\tau(V_0)}(\delta\tilde{n} - \frac{\partial n_{eq}}{\partial V}\Big|_{V_0}\delta\tilde{V})$$

$$\Rightarrow -\imath\omega C\delta\tilde{V} = [-g(V_0 - V_{ion})[1 - \imath\omega\tau(V)]^{-1}\frac{\partial n_{eq}}{\partial V}\Big|_{V_0} - (gn_{eq} + G_{leak})]\delta\tilde{V} + \delta\tilde{I} \tag{33}$$

$$Z^{-1}(\omega) = \frac{\delta \tilde{I}(\omega)}{\delta \tilde{V}(\omega)} = -\imath\omega C + g n_{eq} + G + g(V_0 - V_{ion}) \frac{\partial n_{eq}}{\partial V}\Big|_{V_0} [1 - \imath\omega\tau(V)]^{-1} \tag{34}$$

$$\delta V(t) = \int dt' Z(t-t') \delta I(t') + O[(\delta I)^2] \tag{35}$$

where

$$Z(t) = \int d\omega \tilde{Z}(\omega) e^{-\imath\omega t} \tag{36}$$

1. Stability analysis

The system is stable around the equilibrium position at V_0 if poles of $Z(\omega)$ have $Im(\omega) < 0$.

$$(\imath\omega)^2 C\tau - (\imath\omega)(C + G_\tau) + (G_1 + G_2) = 0 \tag{37}$$

$$G_1 = g n_{eq} + G_{leak} \tag{38}$$
$$G_2 = g \frac{\partial n_{eq}}{\partial V}\Big|_{V_0} (V - V_{ion})$$

$$\Rightarrow \omega = -\imath \frac{(C + G_1\tau) \pm \sqrt{(C + G_1\tau)^2 - 4C\tau(G_1 + G_2)}}{2C\tau} \tag{39}$$

So the stability condition, corresponding to $G_1 + G_2 > 0$ read

$$V_0 > \{V_{ion} - \frac{(n_{eq} + G/g)}{\frac{\partial n_{eq}}{\partial V}\Big|_{V_0}}\} \tag{40}$$

Since $V_{ion} < V_0 < V_{rest}$, stability is given if $V_{ion} < V_{rest}$.
So, defining the two resistors R_1, R_2 and the inductor L,

$$R_1^{-1} = g n_{eq} + G \tag{41}$$
$$R_2^{-1} = g(V_0 - V_{ion}) \frac{\partial n_{eq}}{\partial V}\Big|_{V_0} \tag{42}$$
$$L = R_2 \tau(V_0) \tag{43}$$

one can sketch the equivalent circuit:
In these terms, stability is implied by the condition $R_2 > 0$. If R_2 becomes negative and $|R_2| > |R_1|$ the system becomes unstable. Then, if $V_{ion} < V_{out}$ the equilibrium point is uniquely defined by

$$n(V) = -\frac{G}{g} \frac{V - V_{rest}}{V - V_{ion}} \tag{44}$$

and the system, after a small perturbation, will relax to the equilibrium position, without oscillations, (see fig.1-fig.2).The other case, $V_{rest} < V_{ion}$, might present more than one equilibrium position (e.g. three, the unstable position at V_0 together with two stable solutions above and below this value). (see fig.3-fig.4).

2. Propagating Fronts

We than describe the voltage along the axon, writing

$$C\frac{\partial V}{\partial t} + D\frac{\partial^2 V}{\partial x^2} = -gn(V - V_{ion}) - G_{leak}(V_{rest}) + I \tag{45}$$

$$\frac{dn}{dt} = -\frac{1}{\tau(V)}(n - n_{eq}(V)) \tag{46}$$

The validity of the *wave equation*

$$\frac{\partial^2 V}{\partial x^2} = (\frac{c\tau(V_0)}{D})\frac{\partial^2 V}{\partial t^2} \tag{47}$$

and the existence of a traveling solution for (45) is then justified in the limit where $n_0(V)$ is large enough (saturated regime) so that $\tau(V_0)$ is almost constant, i.e. if the rate is almost constant over the period of a pulse. So, in order for the wave equation to be satisfied, i.e. for the solution to be traveling with constant velocity v the conditions are:

$$\frac{\partial^2 V}{\partial x^2} \text{ is small over a pulse} \tag{48}$$

$$V \simeq V_{ion} \tag{49}$$

Problems for lectures 3 and 4

1. Variation and the average constrain- Entropy.

Let, as before, minimize the *Lagrange multiplier constrained functional*

$$\delta\Big\{\sum_{n\geq 0} P(n)\log P(n) + \lambda_1(\sum_{n\geq 0} nP(n) - \bar{n}) - \lambda_2(\sum_{n\geq 0}(n) - 1)\Big\} = 0$$

The normalization constrain corresponds to

$$\frac{e^{\lambda_1 - 1}}{1 - e^{-\lambda_2}} = 1 \tag{50}$$

So that

$$P(n) = e^{\lambda_1 - 1 - \lambda_2} = (1 - e^{-\lambda})e^{-\lambda n} = \mathcal{N} e^{-\lambda n}$$

The second lagrange multiplier, λ_2 corresponding to the average $\bar{n}$ constrain,

$$\bar{n} = (1 - e^{-\lambda}) \sum_{n \geq 0} e^{-\lambda n} = (1 - e^{-\lambda})[-\frac{\partial}{\partial \lambda} \sum_{n \geq 0} e^{-\lambda n}] = \frac{e^{-\lambda}}{1 - e^{-\lambda}} \tag{51}$$

$$\lambda = \log(\frac{1 + \bar{n}}{\bar{n}}) \tag{52}$$
$$\mathcal{N} = \frac{1}{1 + \bar{n}}$$

2. Entropy considerations

$$S_{max} = -\sum_{n \geq 0} P(n) \log P(n) = \log(1 + \bar{n}) + \bar{n} \log(\frac{1 + \bar{n}}{\bar{n}}) \tag{53}$$

Let $\bar{r}$ to be the *mean spike rate* and $\Delta\tau$ the size of the *time bins*. The maximum entropy per spike is then given by

$$\frac{S_{max}(\bar{n}))}{\bar{n}} = \frac{1}{\bar{n}} \log(1 + \bar{n}) + \log(1 + \frac{1}{\bar{n}}) \tag{54}$$

Consider a fixed value of the *mean spike rate*, (e.g. $\bar{r} = 30 sp./sec$ as in fig.5).

In the limit of $\Delta\tau << 1$

$$\frac{S_{max}(\bar{n}))}{\bar{n}} \simeq 1 + \log(1 + \frac{1}{\bar{n}}) \simeq \log(\frac{1}{\bar{n}}) \to \infty \tag{55}$$

Vice-versa, for $\Delta\tau >> 1$

$$\frac{S_{max}(\bar{n}))}{\bar{n}} \simeq \frac{\log(\bar{n})}{\bar{n}} \to 0 \tag{56}$$

3. Systematic and random errors in the naive estimate of the entropy

Then, words are called n_i and labelled by i. To estimate $\tilde{p} = n_i/N$ and $\tilde{S} - \sum_i \tilde{p}_i \log \tilde{p}_i$ consider

$$S_{real} = -\sum_i \frac{\bar{n}_i}{N} \log(\frac{\bar{n}_i}{N}) \tag{57}$$

$$\tilde{S} = -\sum_i \frac{n_i}{N} \log \frac{n_i}{N} = -\sum_i (\frac{\bar{n}_i}{N} + \delta_i) \log (\frac{\bar{n}_i}{N} + \delta_i) \tag{58}$$

where $\delta_i \equiv (n_i - \bar{n}_i)/N$ This gives

$$\langle \tilde{S} \rangle_{\{n_i\}} \simeq -\sum_i \frac{\bar{n}_i}{N} \log \frac{\bar{n}_i}{N} - \frac{1}{2} \frac{N}{\bar{n}_i} \langle \delta_i^2 \rangle = S_{real} - \frac{1}{2N} (\sum_i 1) \equiv S_{real} - \frac{M}{2N} \tag{59}$$

where the sum counts the M number of *words* and N the number of *observations* and where the second term represent the systematic error. Similarly, for the random error

$$\langle(\tilde{S}-S)^2\rangle \simeq \sum_i \frac{\bar{n}_i}{N^2}(1+\log\frac{\bar{n}_i}{N})^2 \tag{60}$$

Lower bound for the entropy

$$S = -\sum_i p_i \log p_i = \langle \log p_i \rangle \geq -\log(\langle p_i \rangle) \tag{61}$$

Let $P_c = \sum_i p_i^2$, then

$$S \geq -\log P_c \tag{62}$$

follows from the *convex* properties of the logarithm, i.e.

$$\log(\frac{a+b}{2}) > \frac{1}{2}\log(a+b)$$

5. Estimator

To show that the minimum mean square error estimator $\tilde{S}$ is the conditional mean, minimize

$$\chi^2 = \int_{-\infty}^{\infty} p(s|y)(\tilde{s}(y)-s)^2 ds) \tag{63}$$

$$\frac{\partial\chi^2}{\partial\tilde{s}(y)} = 0 \rightarrow \int_{-\infty}^{\infty} p(s|y)2(\tilde{s}(y)ds - s) = 0$$

$$E[\tilde{s}(y)|y]_{s|y} = E[s|y]_{s|y} \tag{64}$$

so

$$\tilde{s}(y) = \int ds\ s\ P(s|y) \tag{65}$$

6. Gaussian signals

Let $y = s + \eta$, where s, η are Gaussian signals. Let's write the minimum mean square error, in the case in which s, η are not correlated. Let

$$p(s) = \frac{1}{\sqrt{2\pi\sigma_s^2}} e^{-\frac{(s-\mu_s)^2)^2}{2\sigma_s^2}} p(\eta) = \frac{1}{\sqrt{2\pi\sigma_\eta^2}} e^{-\frac{(\eta-\mu_\eta)^2)^2}{2\sigma_\eta^2}} \tag{66}$$

and let

$$v = \frac{y-\mu_y}{\sigma_y}, w = \frac{s-\mu_s}{\sigma_s} \tag{67}$$

where $\mu_y = \mu_s + \mu_\eta$ and $\sigma_y^2 = \sigma_s^2 + \sigma_\eta$. The *mean squeared error estimator* is then

$$\tilde{S} = E[s|y] = E[\sigma_s w + \mu_s | v = \frac{y-\mu_y}{\sigma_y}]$$

$$= \sigma_s E[w|v = \frac{y-\mu_y}{\sigma_y}] + \mu_s \equiv \sigma_s(\frac{\sigma_{sy}}{\sigma_s\sigma_y})\frac{(y-\mu_y)}{\sigma_y} + \mu_s \tag{68}$$

S is then linear in y, since the joint probability density $p(w|v)$ is also gaussian with mean and variance given by $v\rho$ and $1-\rho^2$, where $\rho = (\frac{\sigma_{sy}}{\sigma_s\sigma_y})$. The fraction of the power that belong to the signal is then

$$\xi = \frac{\sigma_s}{\sigma_s\sigma_y}.$$

To show that the estimator is , in general, a non linear function of y, write

$$P(s) = \frac{1}{2s_o}e^{-\frac{|s|}{s_o}} \tag{69}$$

$$\tilde{S} = \int \; s \; p(s|y)ds = \frac{1}{2s_o}e^{-(y-s_o)^2}$$

7. Poisson processes

Let's assume now that a neuron will generate spikes via a Poisson process. The rate $r(t) = \bar{r}x^2(t)$ at which the neuron will spike, depends then on the gaussian distributed variable $x(t), \sigma_x^2 = 1$. The estimator is

$$\tilde{S}(t;\{t_i\}) = \int dss(t)\frac{P(\{t_i\}|s(t))P(s|t)}{P(\{t_i\})} \tag{70}$$

$$\int ds \; s(t) \; \frac{1}{N!}r(t_1)...r(t_N)e^{-\int_0^T r(t)dt}\frac{P(s(t))}{P(\{t_i\})} \tag{71}$$

$$\int ds \; s(t) \; \frac{1}{N!}r(t_1)...r(t_N)e^{-\int_0^T r(t)dt}e^{-\frac{T}{4}\int dt\dot{x}^2-\frac{1}{4T}\int dtx^2} \tag{72}$$

In Fourier space, the previous quantity reduces then to

$$\tilde{S}(t;\{t_i\}) = \frac{\sum_{pairings}\prod_{<ij>}\int\frac{d\omega_i}{2\pi}\frac{e^{\imath\omega_i(t_i-t_j)}}{1/4T-T/4\omega_i^2}}{\sum_{pairings}\prod_{<kl>}\int\frac{d\omega_k}{2\pi}\frac{e^{\imath\omega_i(t_k-t_l)}}{1/4T-T/4\omega_k^2}} \tag{73}$$

where i,j are drawn from $\{0,...,2N+1\}$, and k,l from $\{0,...,2N\}$. In the limit here the spikes are sparse, only the pairs in the form t, t_i will contribute, and thus the decoding is linear, i.e:

$$\tilde{S}(t;\{t_i\}) \simeq \sum_i K(t-t_i) \tag{74}$$

Problems for Lectures 5

Fractional frequency response

The two variables $x_i, i = 1,2$, e.g. the voltage and the number of open channels, satisfy

$$\frac{dx_1}{dt} = -Ax_1 + Bx_2 + I \tag{75}$$
$$m\frac{dx_2}{dt} = Cx_1 - Dx_2$$

The first constrain on the coefficients, comes from the low frequency limit argument. In Fourier space:

$$\imath\omega\tilde{x_1} = -A\tilde{x_1} + B\tilde{x_2} + \tilde{I} \tag{76}$$
$$\imath\omega\tilde{x_2} = C\tilde{x_1} - D\tilde{x_2} \tag{77}$$

This gives

$$\Rightarrow \qquad \tilde{Z} = \frac{\tilde{x}_i}{\tilde{I}} = 0 \;\; at\ \omega = 0$$

so that $D = 0$. Also note that the form of the whole frequency response is not fractional, as observed experimentally, but given by

$$|\tilde{Z}| = \frac{\omega}{\sqrt{\omega^4 + (2BC + A^2)\omega^2 + B^2C^2}} \tag{78}$$

Taylor expansion of the previous expression gives

$$|\tilde{Z}| \simeq \frac{1}{BC}\omega - \frac{(2BC + A^2}{2B^2C^2}\omega^3 + .. \tag{79}$$

The plot of the frequency response is in fig.7.

1.Diffusion equation

Let's consider now a diffusive process

$$\frac{\partial c}{\partial t} = D\nabla^2 c + I(t)\delta(\vec{x}) \tag{80}$$

In Fourier space, in one spacial dimension

$$\imath\omega\tilde{c}(\omega, q) = -Dq^2\tilde{c}(\omega, q) + \tilde{I}(\omega) \tag{81}$$
$$\tilde{c}(\omega, q) = \frac{I(\omega)}{(\imath\omega + Dq^2)}$$

$$c(\omega, x) = \int \frac{dq}{2\pi} e^{\imath qx} \frac{I(\omega)}{(\imath\omega + Dq^2)} \tag{82}$$

and the two poles $\pm q_o$ are located at

$$q_o = \frac{1 + \imath}{\sqrt{2}} \sqrt{\frac{\omega}{D}} \tag{83}$$

In the low frequency limit, $\omega \to 0$,

$$c(\omega, x) = \frac{(1 + \imath)}{2^{3/2}\sqrt{D}} \frac{1}{\sqrt{\omega} I(\omega)} \equiv Z(\omega)I(\omega) \tag{84}$$

To explain the square root behavior of the frequency response, consider then the three dimensional case where:

$$c(\omega,\vec{x}) = \int \frac{d\vec{q}}{(2\pi)^3} e^{\imath\vec{q}\vec{x}} \frac{I(\omega)}{\imath\omega + D|\vec{x}|^2} \tag{85}$$

$$= 2\int_0^1 d(\cos\theta) 2\pi\imath \frac{1}{(2\pi)^2} e^{\imath q_o x\cos\theta} \frac{I(\omega)}{2D} \tag{86}$$

$$\frac{\imath}{2\pi}\frac{I(\omega)}{Dx}(e^{\frac{\imath-1}{\sqrt{2}}\sqrt{\frac{\omega}{D}}} - 1)$$

In the low frequency limit, this reduces to the fractional power behavior predicted by the experiments.

$$c(\omega,\vec{x}) \simeq -\frac{1+\imath}{\pi(2D)^{\frac{1}{2}}}\sqrt{\omega}I(\omega) \tag{87}$$

An alternative way of obtaining the $\sqrt{\omega}$ behavior is by including a feedback in the diffusive process.

$$c(\omega) = K(\omega)\Big(I(\omega) + Z(\omega)c(\omega)\Big) \tag{88}$$

In the case of a process K such that $K(\omega) \to const.$ for $\omega \to 0$, then

$$\tilde{Z}'(\omega) \sim \frac{1}{1-\frac{1}{\sqrt{\omega}}} \sim \sqrt{\omega}. \tag{89}$$

2. Lateral inhibition

Imagine then the neurons on a lattice, each receiving input from the outside word, and interacting with its neighbors. In one dimension, consider

$$\frac{dV_n}{dt} = -\frac{1}{\tau}V_n - J(V_{n-1}+V_{n+1}) + I_n, \tag{90}$$

where J positive means that neighbors inhibit one another. In Fourier space:

$$\imath\omega\tilde{V}(\omega,n) = -\frac{1}{\tau}\tilde{V}(\omega,n) - J(\tilde{V}(\omega,n-1)+\tilde{V}(\omega,n+1)) + \tilde{I}(\omega,n) \tag{91}$$

this gives

$$\tilde{Z} = \frac{\tilde{V}}{\tilde{I}} = \frac{1}{\imath\omega + \frac{1}{\tau} + 2J\cos k} \tag{92}$$

In the low frequencies limit k, one find

$$|Z| = \frac{1}{\omega^2 + (\frac{1}{\tau} + 2J(1-\frac{1}{k^2}))} \simeq a + bk^2 \quad -\pi \le k \le \pi \tag{93}$$

In order to obtain optimal encoding we then enforce the fractional power behavior, obtaining an expression for the interactions between neighboring neurons

$$\tilde{Z}^{-1} = \frac{1}{c + 2\pi\sum_m J(m)\cos km} = \sqrt{|k|} \quad c = 2/\sqrt{\pi} \tag{94}$$

and

$$J(m) = \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{1}{\sqrt{|k|}}\cos mkdk = \sqrt{\frac{2\pi}{m}}C(\sqrt{2m}) \tag{95}$$

where C is the *Fresnel cosine integral*,and $C(z) = \int_0^z \cos(\frac{\pi}{2}t^2)dt$; for integer values of z. Similarly, in two spacial dimensions, one write

$$\frac{\partial V_{\vec{r}}}{\partial t} = -\frac{1}{\tau}V_{\vec{r}} - \sum_i J(\vec{r})(V_{\vec{r}-\vec{s}} + V_{\vec{r}+\vec{s}}) + I_{\vec{r}}.$$

The optimal filtering constrain then read

$$\sum_{\vec{s}} J(\vec{s})\cos(\dot{\vec{q}\vec{s}}) = \frac{1}{2\sqrt{q}} \tag{96}$$

$$J(\vec{s}) \sim \frac{1}{|\vec{s}|^{3/2}} \quad for \quad |\vec{s}| \to \infty$$

Problems for lecture 6

1.

Consider to minimize the probability of error. For each x:

$$E = \frac{1}{2} + \frac{1}{2}\int f(x)[P_+(x)(1-f(x)) + P_-(x)f(x)] \tag{97}$$

The choice of $f(x)$ that correspond to the *maximum likelihood decision rule* is

$$f(x) = \Theta(P_+(x) - P_-(x)) \tag{98}$$

where $\Theta(x) = 0 \ for \ x < 0 \ and \ \Theta = 1 \ for \ x > 0$.
Consider then two gaussian distributions for $P_+ \ and \ P_-$ of average μ_1, μ_2 and variance σ, so that

$$E_{min} = \frac{1}{2}\int_{\frac{\mu_1+\mu_2}{2}}^{\infty} P_- + \frac{1}{2}\int_{-\infty}^{\frac{\mu_1+\mu_2}{2}} P_+(x) = 1 - erf\Big(\frac{\mu_1-\mu_2}{2}\Big) \tag{99}$$

In the limit of $|\mu_1 - \mu_2| \to \infty$ one has $E_{min} \to 0$, while, for $|\mu_1 + \mu_2| \to 0$ one find $E_{min} = \frac{1}{2}\Big(1 - \frac{\mu_2-\mu_1}{\sqrt{2\pi\sigma^2}}\Big)$. The other case correspond to two gaussian distributions with the same mean but different variances.

$$P_\pm(x) = \frac{1}{\sqrt{2\pi\sigma_\pm^2}}e^{-\frac{x^2}{2\sigma_\pm^2}}$$

Then, for $\sigma_+ < \sigma_-$

$$E_- = 2\int_{\frac{\alpha}{2}}^{\infty} P_-(x)dx = \frac{1}{\sqrt{2}}(1 - erf(\frac{\alpha}{2\sigma_-})) \tag{100}$$

$$E_+ = 2\int_0^{\frac{\alpha}{2}} P_+(x)dx = \frac{1}{\sqrt{2}}(1 - erf(\frac{\alpha}{2\sigma_+})) \tag{101}$$

as expected these two quantities are not equal.
E is then minimized when we consider the *maximum likelihood solution* according to:

$$\frac{P_+}{P_-} \sim 1 \to \frac{\sigma_1}{\sigma_2} e^{-x^2(\frac{1}{2\sigma_+^2} - \frac{1}{2\sigma_-^2})} \sim 1 \tag{102}$$

this ratio define indeed the quantity α in the previous equation

$$|x| \sim \alpha \equiv \frac{2\sigma_+\sigma_-}{\sqrt{2\sigma_+^2 - 2\sigma_-^2}} \sqrt{\log(\frac{\sigma_+}{\sigma_-})} \tag{103}$$