8.581 Entropy, information and the brain

Problems for Prof. Kardar's lectures.

---

1. A scalar field $m(\mathbf{x}, t)$ evolves according to the equation

$$\frac{\partial m}{\partial t} = m - m^3 + K\nabla^2 m - \nabla^4 m. \tag{1}$$

(a) Find the cost function whose variations lead to this equation.
(b) Linearize the above equation around $m(\mathbf{x}, t) = 0$, and discuss the stability of various Fourier modes for $K > 0$ and $K < 0$. In each case indicate the most unstable mode.
(c) Analyze the stability of the ordered phases by setting $m(\mathbf{x}, t) = 1 + \epsilon(\mathbf{x}, t)$, and linearizing the equation of motion. For what values of K are these phases unstable?

2. Consider complex functions $z_n(t)$ along a linear chain ($n = 1, 2, \cdots$), evolving according to

$$\frac{\partial z_n}{\partial t} = z_n(1 - |z_n|^2) + K_1\left(z_{n+1} + z_{n-1}\right) + K_2\left(z_{n+2} + z_{n-2}\right). \tag{2}$$

(a) Find the cost function whose variations lead to this equation.
(b) Analyze the linear stability of the configuration $\{z_n = 0\}$.
(b) What is the saturation value $\overline{z}$ for the magnitude of z_n?
(c) Set $z_n = \overline{z}\exp(i\theta_n)$, and obtain the evolution equation for the phases $\{\theta_n\}$. Discuss the linear stability of the corresponding Fourier modes.