**8.581: Entropy, information and the brain**

Solutions for Prof. Kardar's Problem Set IAP 1998

**1.**

(a) **The cost function**

$$F[m] = \int dx \, \left\{ \frac{1}{2} m^2(x) - \frac{1}{4} m^4(x) - K|\nabla m|^2 - \frac{1}{2}|\nabla^2 m|^2 \right\},$$

has a local minimum at the stationary solution of the given equation of motion

$$\frac{\partial m}{\partial t} = m - m^3 + K\nabla^2 m - \nabla^4 m.$$

It can be shown through integration by parts to obtain for a variation $\eta(x)$ around $m(x)$, that

$$F[m+\eta] - F[m] = \int dx \, \eta(x) \, \left[ m(x) - m^3(x) + K\nabla^2 m(x) - \nabla^4 m(x) \right].$$

(b) **Linearize around** $m(x) = 0$ to get

$$\frac{\partial m}{\partial t} = m + K\nabla^2 m - \nabla^4 m.$$

Take the Fourier transform in space domain,

$$\frac{\partial m(\vec{q},t)}{\partial t} = \left(1 - Kq^2 - q^4\right) \, m(\vec{q},t),$$

which has the solution $m(\vec{q},t) = m(\vec{q},0)\, e^{-t/\tau(q)} + m(\vec{q}, t=\infty)$ , with $\tau(q) \equiv q^4 + Kq^2 - 1$. The Fourier modes labelled by $q$ are stable if the exponential term vanishes as $t \to \infty$, i.e. $\tau(q) > 0$. Analyzing the roots and the minimum of the polynomial $\tau(q)$, gives the stability condition

$q^2 > -\frac{K}{2} + \sqrt{\frac{K^2}{4} + 1}$.
For $K > 0$, maximum instability is at $q = 0$.
For $K < 0$, maximum instability is at $q^2 = \frac{|K|}{2}$.

(c) **Stability around** $m(\vec{x},t) = 1$.

Linearizing the equation of motion around $m = 1$ gives

$$\frac{\partial \varepsilon(\vec{q},t)}{\partial t} = -2\varepsilon(\vec{q},t) - q^2 K\varepsilon(\vec{q},t) - q^4\varepsilon(\vec{q},t)$$

Same analysis as in (b) yields characteristic times $\tau(q) = 2 - Kq^2 + q^4$. All modes are now stable for $K > 2\sqrt{2}$. For $K < 2\sqrt{2}$, unstable modes occur for

$$-\frac{K}{2} - \sqrt{\frac{K^2}{4} - 2} < q^2 < -\frac{K}{2} + \sqrt{\frac{K^2}{4} - 2}.$$

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**2.**

(a) **The cost function** that leads to the given equation of motion is

$$F[\{z_n\}] = \sum_n (1 + 2K_1 + 2K_2)|z_n|^2 - \frac{1}{2}|z_n|^4 - K_1|z_{n+1} - z_n|^2 - K_2|z_{n+2} - z_n|^2.$$

(Observe that $\partial F/\partial z_n^* = z_n - z_n|z_n|^2 + K_1(z_{n+1} + z_{n-1}) + K_2(z_{n+2} + z_{n-2})$. )

(b) **Linear stability**:

In the linear regime, ignore the high order term in $z_n$. Then, real and imaginary parts do not couple, and each evolve independently. Define $z(k) = \sum_n z_n\, e^{ink}$, where $k$ corresponds to the Fourier modes (of either real or imaginary part). Then,

$$\partial z(k)/\partial t = z(k)\left[1 + 2K_1 \cos k + 2K_2 \cos 2k\right].$$

A configuration around $\{z_n = 0\}$ is stable with respect to long wavelength fluctuations ($k$ small), if $K_1 + K_2 < -1/2$.

(c) If the $|z_n|$ saturate to a uniform $\bar{z}$ at a local energy minimum, we can neglect the last two terms in $F\,\{z_n\}]$, to get

$$F[\{z_n\}] \simeq \sum_n (1 + 2K_1 + 2K_2)|z_n|^2 - \frac{1}{2}|z_n|^4.$$

Upon minimization, we obtain

$$\bar{z}^2 = 1 + 2\,(K_1 + K_2), \quad \text{for} \quad K_1 + K_2 > -1/2.$$

(d) **Setting** $z_n = \bar{z}e^{i\theta_n}$, results in

$$\frac{\partial e^{i\theta_n}}{\partial t} = e^{i\theta_n} - \bar{z}^2 e^{i\theta_n} + K_1\left(e^{i\theta_{n+1}} + e^{i\theta_{n-1}}\right) + K_2\left(e^{i\theta_{n+2}} + e^{i\theta_{n-2}}\right).$$

For $\theta_{n+1} - \theta_n$ small, the equations are linearized to

$$\frac{\partial \theta_n}{\partial t} = K_1\,(\theta_{n+1} - 2\,\theta_n + \theta_{n-1}) + K_2\,(\theta_{n+2} - 2\,\theta_n + \theta_{n-2}).$$

Fourier transforming gives

$$\begin{aligned}
\frac{\partial \theta(k)}{\partial t} &= K_1 \theta(k)\left[e^{ik} - 2 + e^{-ik}\right] + K_2 \theta(k)\left[e^{2ik} - 2 + e^{-2ik}\right] \\
&= \left\{2K_1(\cos k - 1) + 2K_2(\cos 2k - 1)\right\}\theta(k) \\
&\simeq (2K_1 + 8K_2)\,k^2\,\theta(k),
\end{aligned}$$

where in the last step we expanded for $k$ small. The uniform configuration with $|z_n| = \bar{z}$ is stable for $K_1 + 4K_2 < 0$.

Notice the absence of a constant ($k$-independent) term in the last equation which is a consequence of having continuous degrees of freedom (in contrast with the previous problem).

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