The Random Energy Model

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1. Introduction

In this talk, we present a simple model of disordered system which can be solved exactly. The model is defined as following: The energy levels of the whole system are random independent quenched variables. We obtain an analytic expression of the averaged free energy over all the possible choices of the energy levels. To understand how this model is related to more conventional problems, let us start by considering a simple spin glass model. For such a model, the hamiltonian [1] of N interacting Ising spins can be written as

$$\mathcal{H} = -\sum_{\langle ij \rangle} J_{ij}\sigma_i\sigma_j \tag{1}$$

where the bonds J_{ij} are random quenched variables given by a probability law $p(J_{ij})$ and the spins σ_i are either $+1$ or -1.

We can define for this model *the probability $P_1(E)$ that a given configuration $\{\sigma_i^{(1)}\}$ of the spins has the energy E*

$$P_1(E) = \int \cdots \int \prod_{\langle ij \rangle} p(J_{ij})\, dJ_{ij} \, \delta\left(E + \sum_{\langle ij \rangle} J_{ij}\sigma_i^{(1)}\sigma_j^{(1)}\right).$$

When the distribution of bonds is gauge invariant [2], which means that:

$$p(J_{ij}) = p(J_{ij}\sigma_i\sigma_j) \quad \text{with } \sigma_i = \pm 1 \text{ for any } i$$

the probability $P(E)$ does not depend on the spin configuration.

To show that this distribution $P(E)$ has a simple form, let us give two examples:

Example 1 [2, 3]: The spins are on a lattice with a coordination number z and the distribution of bonds is

$$p(J_{ij}) = \tfrac{1}{2}(\delta(J_{ij} - J) + \delta(J_{ij} + J)).$$

Then one has for large N

$$P(E) \sim \exp -\frac{Nz}{4}\left[\left(1 + \frac{2E}{Nz}\right)\log\left(1 + \frac{2E}{Nz}\right) + \left(1 - \frac{2E}{Nz}\right)\log\left(1 - \frac{2E}{Nz}\right)\right].$$

Example 2: The Sherrington–Kirkpatrick model [4] (the S.K. model) where

$$p(J_{ij}) = \sqrt{\frac{N}{2\pi J^2}}\, \exp -\frac{J_{ij}^2}{2J^2}\, N$$

for any pair of spins i and j in the system.

Then

$$P(E) = \frac{1}{\sqrt{\pi N J^2}} \exp - \frac{E^2}{N J^2}.$$

One can define in the same way $P_{1,2}(E_1, E_2)$ as the probability that two given configurations $\{\sigma_i^{(1)}\}$ and $\{\sigma_i^{(2)}\}$ have energies E_1 and E_2.

For the S.K. model, one finds

$$P_{1,2}(E_1, E_2) \sim \exp - \left\{ \frac{(E_1 + E_2)^2 N}{4 J^2 ((N-n)^2 + n^2)} + \frac{(E_1 - E_2)^2 N}{8(N-n) n J^2} \right\}$$

where n is the number of identical spins in the two configurations. This formula shows how the energies E_1 and E_2 of the two configurations are correlated. In the limit ($n \to 0$ or $n \to N$), the two energies have to be equal while when $n = N/2$, the two energies are completely independent and the probability is factorizable

$$P(E_1, E_2) = P(E_1) P(E_2).$$

2. Definition of the random energy model

We consider an abstract system with the following properties:

(1) The system has 2^N energy levels E_i;

(2) The energy levels E_i are distributed randomly with probability distribution

$$P(E) = \frac{1}{\sqrt{N \pi J^2}} \exp - \frac{E^2}{N J^2}; \tag{2}$$

(3) The E_i are independent random variables.

As we saw in the introduction, the two first assumptions are actual features of some spin glass models.

The third assumption is crucial because it allows to solve exactly the model. However it can be seen as a crude approximation to more realistic models since the correlations between the energies of different configurations are neglected.

The partition function Z is given by

$$Z = \sum_{i=1}^{2^N} \exp\{-\beta E_i\}$$

and as usual for disordered systems, one wants to calculate the average free energy F

$$F = - T \langle \log Z \rangle = - T \int \prod_i (P(E_i) \, dE_i) \log[Z(\{E_i\})].$$

3. The moments $\langle Z^n \rangle$ of the partition function

The integer moments of the partition function are customarily the easiest quantities one can calculate for disordered systems.
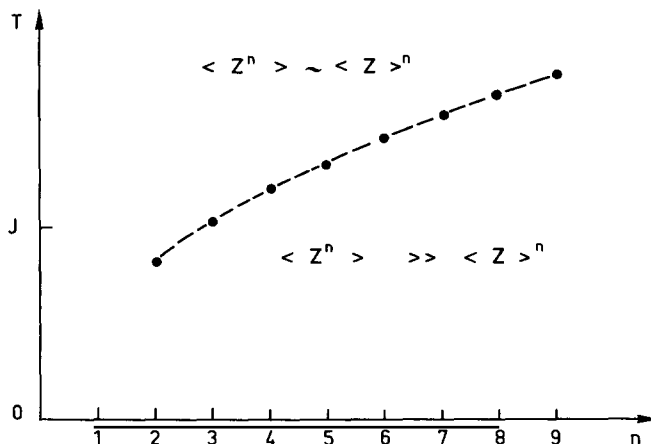
Fig. 1. The integer moments $\langle Z^n \rangle$ as functions of temperature. In the high temperature region $T > T_n$, $\langle Z^n \rangle \sim \langle Z \rangle^n$. In the low temperature region $T < T_n$, $\langle Z^n \rangle \gg \langle Z \rangle^n$.

For this random energy model, one finds that in the thermodynamic limit $(N \to \infty)$, each moment $\langle Z^n \rangle$ changes its behaviour at a critical temperature $T_n = (J/2\sqrt{\log 2})\sqrt{n}$. For large N, the moments are given by

$$\langle Z^n \rangle \sim 2^N \exp\{Nn^2J^2/4T^2\} \qquad \text{if } T < T_n \tag{3}$$

$$\langle Z^n \rangle \sim 2^{nN} \exp\{NnJ^2/4T^2\} \qquad \text{if } T > T_n.$$

As each moment $\langle Z^n \rangle$ has its own critical temperature T_n, the question is what happens for the average free energy at these temperatures T_n.

The simplest approach to compute the average free energy would be to assume that the distribution of Z is concentrated around its average $\langle Z \rangle$

$$\langle \log Z \rangle = \log(\langle Z \rangle) + \left\langle \log\left[1 + \left(\frac{Z}{\langle Z \rangle} - 1 \right) \right] \right\rangle = \log\langle Z \rangle + \sum_{n=2}^{\infty} \frac{(-)^{n+1}}{n} \left\langle \left(\frac{Z}{\langle Z \rangle} - 1 \right)^n \right\rangle. \tag{4}$$

By looking at the behaviour of the moments $\langle Z^n \rangle$ (3), it is obvious however that the series (4) is divergent at any temperature and that formula (4) is useless to calculate the average free energy.

4. The average free energy

It is possible for this random energy model to find the analytic expression of the average free energy when N is large [5]. The details of the calculation are too lengthly to be included here. So we only sum up the results and we indicate in section 5 a simple way to recover the thermodynamic limit.

In the thermodynamic limit, the system has a critical temperature $T_c = J/2\sqrt{\log 2}$.

In the low temperature phase $(T < T_c)$, the system is completely frozen in its ground state and the

average free energy is given by

$$\frac{1}{N}\langle F\rangle = -J\sqrt{\log 2}+\frac{T_c}{2}\frac{\log N}{N}+\frac{T_c}{2N}\log(4\pi\log 2)+(T_c-T)\frac{1}{N}\Gamma'(1)-\frac{T_c}{N}\log\left(\Gamma\left(1-\frac{T}{T_c}\right)\right)$$
$$+O\left(\frac{\log^2 N}{N^2}\right)\tag{5}$$

(where Γ is the gamma function).

In the high temperature phase $(T>T_c)$, the thermodynamic limit is a simple function of temperature but the subdominant terms change their behaviour at all the temperatures $T_{2n+1}=\sqrt{2n+1}T_c$.

If $T_{2n-1}<T<T_{2n+1}$

$$\frac{1}{N}\langle F\rangle = \underline{-(T\log 2+J^2/4T)}+\frac{T}{2N}\left(\frac{\langle Z^2\rangle}{\langle Z\rangle^2}-1\right)+\cdots+(-1)^n\frac{T}{nN}\left\langle\left(\frac{Z}{\langle Z\rangle}-1\right)^n\right\rangle$$
$$-\frac{T^2}{N^{3/2}}\frac{2\sqrt{\pi}}{J}\frac{1}{(T^2/T_c^2+1)}\exp\left\{-\frac{NT^2J^2}{16}\left(\frac{1}{T_c^2}-\frac{1}{T^2}\right)^2\right\}\left[\frac{1}{\sin\frac{1}{2}\pi(T^2/T_c^2+1)}+O\left(\frac{1}{N}\right)\right].\tag{6}$$

In the expressions (5) and (6), the underlined quantities give the thermodynamic limit whereas the other terms describe how this large N limit is reached.

One can notice that the convergence when $N\to\infty$ is very rapid above T_c (because each term $(Z/\langle Z\rangle-1)^n$ in eq. (6) decreases exponentially with N) and is very slow below T_c. At the critical temperature as well as at all the temperatures T_{2n+1}, the subdominant terms are singular. It follows that a high temperature expansion would be very singular as the infinite temperature is the accumulation point of the temperatures T_n.

5. The microcanonical approach

For one sample of the 2^N energy levels, one can define the density $n(E)$ of energy levels:

$$n(E)=\sum_{i=1}^{2^N} x_i\qquad\text{with } x_i=1 \text{ if } E<E_i<E+dE$$
$$=0\text{ otherwise}.$$

One can average this density as well as all its moments. When N is large, one finds

$$\langle n(E)\rangle=\frac{2^N}{\sqrt{\pi NJ^2}}\exp\left(-\frac{E^2}{NJ^2}\right)dE\tag{7}$$

$$\langle n^p(E)\rangle\sim\langle n(E)\rangle^p\qquad\text{if }\langle n(E)\rangle\gg 1$$
$$\sim\langle n(E)\rangle\qquad\text{if }\langle n(E)\rangle\ll 1.\tag{8}$$

From (7) and (8), it follows that
- if $|E|>NJ\sqrt{\log 2}$, the average density $\langle n(E)\rangle$ is much smaller than 1. This means that, for almost all the samples, there is no energy level in this region;
- if $|E|<NJ\sqrt{\log 2}$, the average density $\langle n(E)\rangle$ is very large and the fluctuations of the number of
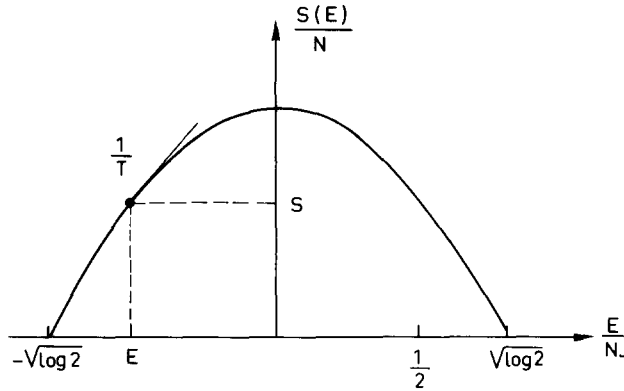
Fig. 2. The entropy $S(E)$ as a function of the energy E.

levels between E and $E + dE$ are of order $\langle n(E)\rangle^{1/2}$. Therefore almost all the samples have $n(E) \sim \langle n(E)\rangle$.

So the entropy $S(E)$ is for almost all the samples

$$S(E) \sim \log\langle n(E)\rangle = N \log 2 - E^2/NJ^2 \quad \text{if } |E| < NJ\sqrt{\log 2}.$$

To obtain the free energy at a given temperature T, one has to find the point of the curve $S(E)$ where the slope is $dS/dE = 1/T$. The energy and the entropy of the system at temperature T are the coordinates of this point.

The critical temperature T_c is the inverse of the slope dS/dE at the edge of the density of states. For $T < T_c$, the system is frozen in its ground state and the energy is $E = -NJ\sqrt{\log 2}$.

Now, we can make two remarks:

a) It seems that the moments $\langle n^p(E)\rangle$ are simpler quantities than the moments $\langle Z^n\rangle$: all the moments $\langle n^p(E)\rangle$ change their behaviour at the same energy $-NJ\sqrt{\log 2}$. It is then natural to think that this energy is the energy of the system at the critical temperature. On the contrary, each moment $\langle Z^n\rangle$ changes its behaviour at its own critical temperature T_n and hence it is less easy to see what is the critical temperature of the average free energy.

b) From (5) and (6), one can find the specific heat C:

$$C/N = 0 \quad \text{if } T < T_c$$

$$C/N = J^2/2T^2 \quad \text{if } T > T_c.$$

The specific heat has a discontinuity at T_c and is zero in the whole low temperature phase.

One can compare this result with the specific heat C' calculated for a ferromagnetic system using the mean field theory. C' has also a discontinuity at the critical temperature but in the ferromagnetic case, it vanishes in the whole high temperature phase.

6. The effect of a uniform magnetic field

To calculate the magnetic susceptibility for this random model, one has to define the probability distribution of the energy levels in presence of a uniform magnetic field H.
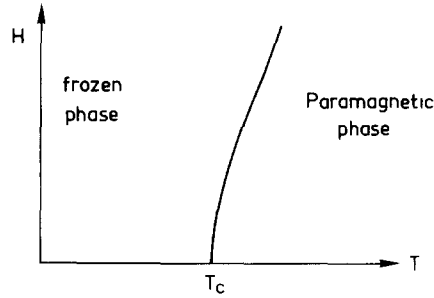
Fig. 3. The phase diagram of the random energy model in a uniform magnetic field H.

If we start again with the spin glass model defined by the equation (1), we find that in a uniform magnetic field H, the probability distribution $P(E)$ depends on the magnetization M of the configuration:

$$P(E) = \frac{1}{\sqrt{\pi J^2 N}} \exp -\frac{(E + MH)^2}{NJ^2} \tag{9}$$

(in the case of example 2).

The random energy model is always a system of 2^N random independent energy levels but among these 2^N energy levels, $C_N^{(N+M)/2}$ have a magnetization M and are distributed with probability distribution (9).

Using the same arguments as in section 5 (the true density of states is equal to its average when the average is large and to zero when the average is small), one can calculate the free energy of the random energy model in a magnetic field H. One finds that for any value of H, there is a transition temperature (fig. 3). In the low temperature phase, the free energy is independent of the temperature. This means that the system is completely frozen in its ground state.

From the free energy, we can obtain the susceptibility when $H = 0$:

$$\chi = 1/T \qquad \text{if } T > T_c$$
$$\chi = 1/T_c \qquad \text{if } T < T_c.$$

We can notice here that although the system is completely frozen in the low temperature phase and has no thermal fluctuations, the response function χ does not vanish.

7. Conclusion

This random energy model may be the simplest model of disordered systems. It gives a simplified picture of a glassy transition: the system becomes completely frozen below the critical temperature. Many of its properties are very similar to those of more conventional spin models: the corrections of order $\log N/N$ to the thermodynamic limit for the ground state energy are to be compared with the $1/N$ [4], or $1/\sqrt{N}$ [6] corrections conjectured for the S.K. model; the same behaviour of the moments $\langle Z^n \rangle$ can be observed for the S.K. model [7]; the constant susceptibility at low temperature has been suggested by Parisi [8] for the S.K. model.

This random energy model shows that looking at the distribution of the level density may be simpler than using the moments $\langle Z^n \rangle$ of the partition function. It can also be seen as a first approximation to any spin glass model. For a large class of models, one can find very easily the probability distribution $P(E)$ and solve the corresponding random energy models. We studied within this approximation the S.K. model with a non symmetric distribution of bonds and we found the same kind of phase diagram that the one predicted by Sherrington and Kirkpatrick [4]. We also used this approximation to calculate the ground state energy for spin glass models with short range interactions on finite dimensional lattices. The results do not differ from the true values by more than 10%.

Starting from the exact solution of this random energy model, one can envisage a systematic inclusion of the correlations between energy levels in order to approach the usual spin glass models.

Acknowledgments

References

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