Spontaneous Symmetry Breaking: General

Spontaneous symmetry breaking is a very common occurrence in many-body systems. Ordinary crystals break translation symmetry down to a discrete subgroup. Ferromagnets break rotational symmetry. In these and many other cases, the stable solutions of the dynamical equations, which govern the system, exhibit less symmetry than the equations themselves.

Superfluidity and superconductivity are also closely associated with spontaneous symmetry breaking, but of a more subtle, intrinsically quantum-mechanical kind. In superfluids – the classic case being liquid He^4 at low temperatures – the symmetry that is broken is the $U(1)$ phase symmetry associated with conservation of He^4 atom number. In superconductors – the classic case being bad metals at low temperatures – the symmetry that is broken is a local (gauged) symmetry, associated to electron number, to which photons respond.

Several cases of spontaneous symmetry breaking are important within the standard model. Two are particularly outstanding.

The approximate chiral symmetry $SU_L(2) \times SU(2)_R$ of QCD, under independent unitary transformations among the left-handed u_L, d_L and the right-handed u_R, d_R helicity states, was the first case to be analyzed deeply, principally by Nambu (in pre-quark days, using a rather different language!). This symmetry is not exact, even within QCD, because it is violated by the non-zero masses of u, d, which flip helicity. Those masses are quite small, however. Quantitatively, the symmetry breaking $SU(2)_L \times SU(2)_R \rightarrow SU(2)_{L+R}$ is predominantly spontaneous. A rich, useful theory of pions and the interactions at low energies follows from these ideas. The symmetry breaking can also be demonstrated directly, by numerical solution of the equations of QCD (lattice gauge theory). In this case the broken symmetry is global[1], similar to superfluidity.

The gauge symmetry $SU(2) \times U(1)$ is postulated in our theory of electroweak interactions. We must, however, avoid the massless gauge bosons that unbroken gauge symmetry seems to imply[2]. This difficulty is overcome by breaking the symmetry spontaneously. In this case, with gauge symmetry front and center, the mechanism is similar to superconductivity.

The full particle physics models have extra complications, which can tend to obscure the basic underlying mechanisms, especially for beginners. Here I will present the basic principles as simply as possible, and simply sketch how they operate in more complicated situations.

[1] Actually the chiral symmetry breaking of QCD also breaks electroweak $SU(2) \times U(1)$. The effect of this gauge symmetry breaking, however, is obscured by the much larger breaking associated with the Higgs field condensation.

[2] Actually unbroken gauge symmetry does not necessarily imply massless vector bosons, as we learn from QCD. There are deep connections, amounting almost to identity, between the ideas of gauge confinement and gauge symmetry "breaking", but I will not plumb those depths here. When the gauge couplings are weak, it is appropriate and fruitful to treat the interplay of symmetry breaking and gauge fields perturbatively, and that is what I'll do in this course. However I cannot resist mentioning a profound wisecrack of mine, that it is much more accurate to speak of gauged symmetry breaking, than of gauge symmetry breaking.

In a separate, subsequent note I'll spell it out more fully for electroweak $SU(2) \times U(1)$ breaking.

Global U(1) Model (Superfluid)

We consider a complex scalar field ϕ, with Lagrangian density

$$\begin{aligned} \mathcal{L} &= \frac{1}{2}\partial^\mu \phi^* \partial_\mu \phi - V(\phi) \\ V(\phi) &= -\frac{\mu^2}{2}\phi^*\phi + \frac{\lambda}{4}(\phi^*\phi)^2 \end{aligned} \tag{1}$$

It is invariant under the phase symmetry

$$\phi \rightarrow e^{i\alpha}\phi \tag{2}$$

(Note, however, that we cannot allow space-time dependence in α.)

The minimum energy solutions of Eqn. (1) will be constant is space and time, with

$$\begin{aligned} \langle\phi\rangle &= ve^{i\eta} \\ v^2 &= \frac{\mu^2}{\lambda} \end{aligned} \tag{3}$$

Here I have put bra-kets around the value of ϕ, to indicate that there are states in the quantum theory that realize the symmetry breaking. In this course we'll proceed heuristically, but there is no great difficulty in constructing the quantum theory that realizes our classical anticipations, at least perturbatively in four space-time dimensions[3].

Energy minimization does not determine the phase η. Indeed, if a minimum energy state violates a symmetry of the equations, then its "symmetry transformed" partners will be different states with the same energy. In other words, we will have a manifold of degenerate states. That is what we have here, with the different choices of η. Gentle interpolation among these states, in space and time, will bring in extra gradient energy, but not local bulk contributions. In the limit of long wavelengths, $k \rightarrow 0$, the energy will go to zero. Given the dispersion relation of relativistic particles, $E^2 = k^2 + M^2$, we expect that the quanta of these modes will have zero mass. They are the celebrated Nambu-Goldstone bosons.

The mathematics bears out that expectation. Let us write

$$\phi = (v + \rho)e^{i\sigma/v} \tag{4}$$

[3] In lower space-time dimensions perturbation theory sometimes breaks down due to accumulation of long-wavelength fluctuations (*infrared* catastrophe).

The kinetic term is then

$$\frac{1}{2}\partial^\mu\rho\partial_\mu\rho + \frac{1}{2}(1+\frac{\rho}{v})^2\partial^\mu\sigma\partial_\mu\sigma \tag{5}$$

For ρ, it is completely conventional. For σ, the leading terms is conventional, while the subsequent terms involving ρ/v represent gradient interactions, which are small corrections as long as the expectation v is much larger than its fluctuation ρ.

The potential does not involve σ at all. We have

$$V(\rho) \;=\; -\frac{\mu^2}{2}(v+\rho)^2 \;+\; \frac{\lambda}{4}(v+\rho)^4 \;=\; -\frac{\mu^4}{4\lambda} + \mu^2\rho^2 + \mu\sqrt{\lambda}\rho^3 + \frac{\lambda}{4}\rho^4 \tag{6}$$

The constant term does not appear in the equations of motion, so we can throw it away. If, however, we were including gravity in our treatment, this term would contribute (negatively) to the density of empty space, also known as the cosmological term or, recently, as dark matter. The quadratic term corresponds to mass $\sqrt{2}\mu$ for the ρ quanta, and the remaining terms some non-linear self-interactions. The σ field is of course massless, as we anticipated on physical grounds.

In constructing the quantum field theory, even perturbatively, it seems necessary to work with basic interactions that are polynomial. Thus we should feel pangs of conscience in using Eqn. (4). We really should use a linearized form, carry out the renormalization, and then use symmetry on the result. This is possible, although somewhat clumsy. Fortunately, "It is more blessed to ask forgiveness than permission."

Gauged U(1) Model (Superconductor)

If we add in a gauge field, so as to make the $U(1)$ phase symmetry local, our Lagrangian becomes

$$\begin{aligned} \mathcal{L} &= -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + \frac{1}{2}\nabla^\mu\phi^*\nabla_\mu\phi - V(\phi) \\ V(\phi) &= -\frac{\mu^2}{2}\phi^*\phi + \frac{\lambda}{4}(\phi^*\phi)^2 \\ \nabla_\mu\phi &= (\partial_\mu + igA_\mu)\phi \end{aligned} \tag{7}$$

Inserting Eqn. (4) into Eqn. (7), we get a potential exactly as before, while the kinetic term boils down to

$$\frac{1}{2}\partial^\mu\rho\partial_\mu\rho + \frac{1}{2}(gvA_\mu + \partial_\mu\sigma)^2(1+\frac{\rho}{v})^2 \tag{8}$$

together with the Maxwell term. The second term is unconventional, and perhaps rather frightening at first sight, but it's actually quite easy to handle, at least heuristically. We simply define a new gauge field

$$A'_\mu \;\equiv\; A_\mu + \frac{1}{gv}\partial_\mu\sigma \tag{9}$$

and note that the Maxwell term for A' has exactly the same form as that for A, since the "gauge-like" gradient term cancels. At this point the σ field has departed from the dynamics: It's been "eaten".

Upon dropping the primes, for ease of notation, the unconventional term becomes

$$\frac{g^2 v^2}{2} A^\mu A_\mu (1 + \frac{\rho}{v})^2 \tag{10}$$

Eqn. (10) suggests, on the face of it, a mass term of magnitude

$$M_A \equiv gv \tag{11}$$

together with some nonlinear interactions, from the terms involving ρ, ρ^2.

To cement the interpretation of the mass term, let's consider the equations of motion we get, by varying with respect to A. We have the usual Maxwell form, with an added contribution, *viz.*

$$0 = -\partial^\mu F_{\mu\nu} + M_A^2 A_\nu = -\partial^\mu (\partial_\mu A_\nu - \partial_\nu A_\mu) + M_A^2 A_\nu \tag{12}$$

Now applying ∂^ν to Eqn. (12), we find

$$M_A^2 \partial^\nu A_\nu = 0 \tag{13}$$

With $\partial^\nu A_\nu = 0$, Eqn. (12) simplifies to

$$(-\partial^2 + M_A^2) A_\nu = 0 \tag{14}$$

We now recognize that each component of A is a massive field with the common mass M_A. They are not all independent, due to Eqn. (13). In fact we have three degrees of freedom, as we should (for massive spin 1).

More Complex Situations

In more complicated situations, we can meet degeneracy manifold of larger dimension, starting gauge groups of larger dimension, subgroups of which may remain unbroken. The general rule, which should seem quite plausible from the preceding, is that if we have a degeneracy manifold of dimension d, for a broken global symmetry, we will have d massless Nambu-Goldstone bosons. If we have local gauge symmetry $G \to H$ in the equations and their solution, and a degeneracy manifold of dimension d, we expect that $\mathrm{Dim}\,(G) - \mathrm{Dim}\,(H)$ of the Nambu-Goldstone bosons get eaten, while the remaining $d - \mathrm{Dim}\,(G) + \mathrm{Dim}\,(H)$ are physical massless quanta. Of course, this quantity must be non-negative. Additional degrees of freedom in our scalar fields, that correspond to motion off the degeneracy manifold, will be ordinary, massive fields.