

Units and Magnitudes (lecture notes)

This lecture has two parts. The first part is mainly a practical guide to the measurement units that dominate the particle physics literature, and culture. The second part is a quasi-philosophical discussion of deep issues around unit systems, including a comparison of atomic, particle ("strong") and Planck units. For a more extended, profound treatment of the second part issues, see arxiv.org/pdf/0708.4361v1.pdf.

Because special relativity and quantum mechanics permeate modern particle physics, it is useful to employ units so that $c = \hbar = 1$. In other words, we report velocities as multiples the speed of light c , and actions (or equivalently angular momenta) as multiples of the rationalized Planck's constant \hbar , which is the original Planck constant h divided by 2π .

In classical physics one usually keeps separate units for mass, length and time. I invite you to think about why! (I'll give you my take on it later.)

To bring out the "dimensional" features of particle physics units without excess baggage, it is helpful to keep track of powers of mass M , length L , and time T without regard to magnitudes, in the form

$$\begin{aligned} [c] &= \frac{L}{T} \\ [\hbar] &= \frac{ML^2}{T} \end{aligned}$$

When these are both set equal to 1, the M, L, T system collapses to just one independent dimension. So we can - and usually do - consider everything as having the units of some power of mass.

Thus for energy we have

$$E = \frac{ML^2}{T^2} = M [c]^2$$

while for momentum

$$P = \frac{ML}{T} = M [c]$$

and for length

$$L = \frac{[\hbar]}{M [c]}$$

so that energy and momentum have the units of mass, while length has the units of inverse mass.

In practice one usually reports quantities using powers of energy, rather than mass. Thus one commonly says that the proton mass is approximately 1 GeV (1 giga-electron volt), which is the energy gained by an electron when subjected to a 10^9 volt electric field gradient. Strictly speaking, in conventional units, what this means is that the proton mass is $m_p \sim 1 \text{ GeV} / c^2$.

The numerical conversion factors, in cgs units, are

$$c \approx 3 \times 10^{10} \frac{\text{cm.}}{\text{sec.}}$$
$$\hbar \approx 10^{-27} \text{ erg} - \text{sec.}$$

and for electron volts

$$1 \text{ eV} \approx 1.6 \times 10^{-12} \text{ erg}$$

Two most important masses, reported in GeV, are

$$m_e = .511 \text{ MeV} \approx .5 \times 10^{-3} \text{ GeV}$$

$$m_p = 938 \text{ MeV} \approx 1 \text{ GeV}$$

The ratio $m_e / m_p \approx 1/2000$ is basic to the structure of the world, yet remains deeply mysterious, as we'll see.

It is instructive, and mildly entertaining, to express the other basic cgs units in GeV. We have, using the above conversion formulae,

$$L = \frac{[\hbar] [c]}{E}$$

and hence

$$\begin{aligned} \text{cm.} &= \frac{\text{cm.}}{\hbar c} \\ &= \frac{\text{cm.}}{3 \times 10^{10} \frac{\text{cm.}}{\text{sec.}} \cdot 10^{-27} \text{ erg} - \text{sec.} \cdot \frac{\text{eV}}{1.6 \times 10^{-12} \text{ erg}}} \\ &\approx \frac{1}{2 \times 10^{-5} \text{ eV}} \\ &= \frac{1}{2 \times 10^{-14} \text{ GeV}} \end{aligned}$$

Conversely, of course, 1 GeV corresponds to an inverse length $(2 \times 10^{-14} \text{ cm.})^{-1}$.

That length has an important physical interpretation. Since $m_p \sim 1 \text{ GeV} / c^2$, the Compton radius of the proton, i.e.

$$r_p = \frac{\hbar}{m_p c}$$

which is the unique length we can form from m_p , c , and \hbar , is nearly the same as the unique length we can form from 1 GeV, c , and \hbar . Thus $2 \times 10^{-14} \text{ cm.}$ is the characteristic size protons acquire simply due to the "quantum kinematics" of uncertainty, when relativistic effects are in play. It corresponds to extrapolating the non-relativistic de Broglie wavelength, which is inversely proportional to velocity, to the limiting velocity c .

It is a profound consequence of QCD that this is also the characteristic *geometric* size of a proton, over which it has significant internal structure.

By contrast the electron, as far as we know, is an accurate point particle, with zero geometric size. (We'll be ready to discuss these matters in a much more sophisticated way later, as the course develops.) The electron's Compton radius is, of course, roughly 2000 times larger than the proton's, in inverse proportion to their masses.

Since

$$\text{sec.} = \frac{\text{cm}}{c}$$

we readily find

$$\text{sec.} \sim \frac{1}{10^{-24} \text{ GeV}}$$

and conclude that 10^{-24} sec. is a characteristic time for protons. It is the time it takes for light to traverse a proton's irreducible quantum fuzziness, or alternatively its intrinsic structure.

Now I'd like to take up a few occasionally confusing issues around electromagnetic units. The fine structure constant in atomic theory is usually written as

$$\alpha = \frac{e^2_{\text{Coulomb}}}{\hbar c} \approx \frac{1}{137}$$

However this is *not* the form we commonly use in quantum field theory or particle physics, where instead you'll find

$$\alpha = \frac{e^2_{\text{rational}}}{4\pi\hbar c} \approx \frac{1}{137}$$

What drives this cultural difference is that in atomic physics it is convenient to have the Coulomb potential between two electrons (say) be e^2/r , whereas in quantum field theory we want simple Lagrangians and Feynman rules. So the atomic physicists want Gauss' law to be

$$\nabla \cdot E = 4\pi\rho$$

while our crowd wants

$$\nabla \cdot E = \rho$$

(Don't even think about bringing in monstrosities like the dielectric permeability and magnetic permittivity of vacuum. I shudder even to mention them.)

The atomic (or gaussian) fields are larger, and the atomic (or gaussian) charges are smaller, by a factor $\sqrt{4\pi}$ relative to their rationalized (or natural) units.

We get to naturalized units in a different way, in SI conventions, by setting the above-mentioned monstrosities equal to 1.

It is interesting to note that $e_{\text{rational}} \approx 1/3$ is *not* a particularly small number. That fact has important, positive implications for the prospects of unified field theory.

The little dictionary for translating units that we've developed so far is already adequate for almost all practical purposes. As an aspiring particle physicist, you should internalize all of it.

One more issue that comes up is the units for magnetic field strengths, which are commonly quoted in gauss (or tesla, with 1 Tesla = 10^4 Gauss). As a useful and

illustrative exercise, let us translate the fundamental definition

$$1 \text{ gauss} = 10^{-4} \text{ kg. / Coulomb-sec}$$

and the value of the electron charge

$$e = 1.6 \times 10^{-19} \text{ Coulomb}$$

into our GeV units.

We have, combining the two preceding equations,

$$\text{gauss} = (10^{-4} \times 1.6 \times 10^{-19} / e) (\text{kg. / sec.})$$

Now we express

$$\text{kilogram} = 10^3 \times 6 \times 10^{23} \times .94 \text{ GeV}$$

$$\text{second} = \text{cm.} / 3 \times 10^{10} \text{ cm./ sec}$$

$$\text{cm.} = 2 \times 10^{-14} / \text{GeV}$$

(In the first of these, we've inserted that a gram contains Avogadro's number of protons, which of course is another important number to know.)

and combine to get

$$\begin{aligned} \text{kg./sec.} &= (10^{16} \times 6 \times .94 / 3) \times (2 \times 10^{-14}) \text{ GeV}^2 \\ &= 3.8 \times 10^2 \text{ GeV}^2 \end{aligned}$$

And now we use $e = \sqrt{4\pi\alpha}$
 $= .3$

to evaluate the prefactor of kg./ sec. as 5.3×10^{-23} .
 Putting it together, we find

$$\text{gauss} = 20 \times 10^{-21} \text{ GeV}^2 = 2 \times 10^{-2} \text{ eV}^2$$

for the (rationalized) gauss.

As an application, we note that

$$(10^{13} \text{ gauss})^2 \sim (m_e)^4$$

so a magnetic field of that magnitude concentrates an electron's mass worth of energy into a volume equal to the electron's Compton wavelength cubed. We might expect interesting things, including specifically electron-positron pair creation, to start happening when there are magnetic fields that large in play.

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Now I'd like to step back and get a larger perspective on what we're doing here, by comparing our "particle physics" units with two other popular systems of units, namely atomic and Planck (or Stoney) units.

(This discussion is basically a digest of one part of my paper "Fundamental Constants", arxiv.org/pdf/0708.4361v1.pdf, mentioned earlier, which I highly recommend!)

Atomic units are based on putting

$$e_{\text{Coulomb}} = m_e = \hbar = 1$$

In other words, we report charge as a multiple of the electron's charge, mass as a multiple of the electron's mass, and angular momentum as a multiple of twice the electron's intrinsic angular momentum.

The great virtue of these units is that it removes all dimensional factors from the non-relativistic Schrödinger equation governing electrons that interact through Coulomb forces with infinitely massive point nuclei and with each other. In principle we can therefore, within this framework, get the energy for different positions of nuclei (with specified charges) interacting with various numbers of electrons. Then by minimizing the energy with respect to nuclear positions we determine the sizes and shapes of molecules - all in terms of pure numbers!

In these atomic units the unit of length is

$$l_a = \frac{\hbar^2}{e_C^2 m_e} = \alpha^{-1} \frac{\hbar}{m_e c}$$

This translates to 5.3×10^{-9} cm. , or roughly half an Ångstrom. It is, not coincidentally, the Bohr radius of hydrogen. It is also equal to the Compton radius of the electron divided by the fine structure constant α .

Since the "relativistic quantum kinematic" size of the electron is $1/137$, in basic units, we can try to include it perturbatively.

In atomic units the unit of velocity is

$$v_a = \frac{e^2 C}{\hbar} = \alpha c$$

Thus the speed of light is 137 in these units, and we can try to do perturbation theory in $1/c$ (nonrelativistic limit).

Atomic units are, for the profound reasons we've just identified, ideally adapted to stereochemistry, and extremely practical for use throughout atomic and molecular physics.

At the other end of practicality spectrum we have the units introduced by Planck at the dawn of quantum theory. Planck's idea was that Newton's gravitational constant G , the speed of light c , and his new constant \hbar

provided three "absolute" units. Absolute units are units that appear so prominently and cleanly in basic laws of physics, that we should be confident (according to Planck) that they have the same value everywhere. Using them we could, for example, convey the length corresponding to a meter to a friend in Andromeda just by sending a number - we needn't send a sample, or anything material.

In modern speculative physics Planck units have a more particular motivation. If we think that a truly fundamental theory should be based on unifying gravity with quantum mechanics and special relativity, then that theory should produce the measured values of physics parameters, such as the electron or proton masses, as pure numbers. Indeed, following this line of thought, dimensional analysis suggests - as a default assumption - that fundamental quantities should be pure numbers of order unity in Planck units.

That naive idea doesn't work out very well. It is instructive to quantify the challenge.

Since

$$[G] = \frac{L^3}{MT^2}$$

the Planck units of length, mass, and time are

$$l_{\text{Pl.}} = \sqrt{\frac{G\hbar}{c^3}} \approx 1.6 \times 10^{-33} \text{ cm.}$$

$$m_{\text{Pl.}} = \sqrt{\frac{\hbar c}{G}} = 2 \times 10^{-5} \text{ gm.}$$

$$t_{\text{Pl.}} = l_{\text{Pl.}}/c = 5 \times 10^{-44} \text{ sec.}$$

or in particle units

$$m_{\text{Pl.}} = l_{\text{Pl.}}^{-1} = t_{\text{Pl.}}^{-1} \sim 1.2 \times 10^{19} \text{ GeV}$$

Evidently $m_{\text{Pl.}} \gg m_p$, so if we regard the Planck units as fundamental we've got the big challenge of understanding why protons are so light.

We'll be rising to that challenge!

We also have the superficially similar-looking question, in this framework, of why electrons, quarks, etc. are so light. That is what is usually called "The Hierarchy Problem". In this course we'll eventually frame the hierarchy problem in a sharper and more sophisticated way, but not solve it. Next semester I'll be discussing supersymmetry, which might help.

Finally let me mention that prior to Planck, and prior to quantum theory, the remarkable physicist George Johnstone Stoney introduced a system of absolute units closely related to Planck's more famous one. (Stoney also first named and pioneered the study of electrons, long prior to Thomson's definitive discovery). Stoney proposed a system of units based on e , c , G . Since the fine-structure constant relates e , c , and \hbar , it is easy to

transform between Planck and Stoney units, and their length, mass, and time scales are not terribly different numerically. Stoney units pose the interesting question whether Planck's constant might be *derived* as a pure number times e^2/c .

In the spirit of atomic (Bohr) and Planck units, which prioritize atomic and quantum gravitational physics, respectively, it is natural to consider absolute units based on nuclear or strong-interaction physics, based on m_p , c , and \hbar . Actually that's what we've done, in effect, with our particle physics units!

All we need to do, to implement the absolute "strong" units, is to substitute $\text{GeV} = m_p / .938$ (and then set $m_p = 1$).