MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Department of Physics
String Theory (8.821) – Prof. J. McGreevy – Fall 2007

Problem Set 3
2d CFT

Due: Thursday, October 11, 2007 at 11:00 AM, in the 8.821 lockbox.
Reading: Ginsparg CFT notes §1,2,3,6. Try §4. Polchsinski, Chapter 2. GSW, Chapter 3 (note that the CFT is hidden in §3.2, entitled 'BRST quantization').

1. **Warmup.** Compute the scaling dimension of the operator $\mathcal{O}_k \equiv : e^{ikX} :$ in the free boson theory whose stress tensor is $T(z) = -\frac{1}{\alpha'} : \partial X \partial X(z) :$, using the $T\mathcal{O}_k$ OPE.

2. **Linear dilaton CFT.**

   The *linear dilaton theory* is a 2d CFT made from a free boson in the presence of a 'background charge.' This means that the boson X has some funny coupling to the worldsheet gravity, which can be described by a linear dilaton term in the action $S_\Phi = \int d^2\sigma \Phi(X) R^{(2)}$, $\Phi(X) = QX$, Q is a constant. On a flat worldsheet, the quantization of X proceeds as before. In that case, the only difference from the ordinary free boson is that the stress tensor (which is sensitive to how the theory is coupled to gravity) has the form

   $$T_Q = -\frac{1}{\alpha'} : \partial X \partial X : + V \partial^2 X$$

   (where $V \sim Q$).

   [Optional: relate V to Q.]

   (a) Verify that T_Q has the right OPE with itself to be the stress tensor for a CFT. Compute the Virasoro central charge for the linear dilaton theory.

   (b) Compute the scaling dimension of the operator $: e^{ikX} :$ in the linear dilaton theory.

   Extra stimulation: Can you interpret the result of (b) in terms of a target space effective action?

3. **The stress tensor is not a conformal primary if $c \neq 0$.** [1]

[1] I got this problem from Robbert Dijkgraaf.

(a) For any 2d CFT, use the general form of the TT OPE to show that the transformation of T under an infinitesimal conformal transformation $z \mapsto z + \xi(z)$ is

$$-\delta_\xi T(w) = (\xi\partial + 2\partial\xi)T(w) + \frac{c}{12}\partial^3\xi. \quad (1)$$

(b) Consider the *finite* conformal transformation $z \mapsto f(z)$. Show that (1) is the infinitesimal version of the transformation law

$$T_{zz}(z) = (\partial f)^2 T_{ff}(f(z)) + \frac{c}{12}\{f, z\}$$

where

$$\{f, z\} \equiv \frac{\partial f \partial^3 f - \frac{3}{2}(\partial^2 f)^2}{(\partial f)^2}$$

is called a *Schwarzian derivative.*

[Optional: verify that this extra term does the right thing when composing two maps $z \to f(z) \to g(f(z))$.]

(c) Given that the conformal map from the cylinder to the plane is $z = e^{-iw}$, show that (b) means that

$$\left(T_{\text{cyl}}(w) - \frac{c}{24}\right)(dw)^2 = T_{\text{plane}}(z)(dz)^2.$$

Use this relation to show that the Hamiltonian on the cylinder

$$H = \int \frac{d\sigma}{2\pi} T_{\tau\tau}$$

is

$$H = L_0 + \tilde{L}_0 - \frac{c + \bar{c}}{24}.$$

Comment: After all this complication, the result has a very simple physical interpretation: when putting a CFT on a cylinder, the scale invariance is spontaneously broken by the fact that the cylinder has a *radius*, *i.e.* the cylinder introduces a (worldsheet) length scale into the problem. The term in the energy extensive in the radius of the cylinder (and proportional to c) is actually experimentally observable.

4. **Constraints from Unitarity.** Show that in a unitary CFT, $c > 0$, and $h \geq 0$ for all primaries. Hint: consider $\langle\phi|[L_n, L_{-n}]|\phi\rangle$.

5. $SU(2)_1$ **current algebra from a circle.**

Consider the closed bosonic string compactified on a circle of radius $R = \sqrt{\alpha'}$. In lecture 4 all kinds of ridiculous claims were made about this theory. Here we will study the CFT describing the strings on this circle and verify that there is in fact an $SU(2)_L \times SU(2)_R$ gauge symmetry involving winding modes. We'll focus on the holomorphic (L) part; the antiholomorphic part will be identical.

Label the circle coordinate $X^{25} \equiv X \sim X + 2\pi R$. Define

$$J^\pm(z) \equiv : e^{\pm 2iX(z)/\sqrt{\alpha'}} :, \qquad J^3 \equiv i\sqrt{\frac{2}{\alpha'}}\partial X(z).$$

(a) Show that $J^3, J^\pm$ are single-valued at the 'self-dual radius' $R = \sqrt{\alpha'}$.

(b) At the self-dual radius, do $J^\pm, J^3$ have the right conformal dimension to create physical string states which are massless?

(c) Defining $J^\pm \equiv \frac{1}{\sqrt{2}}(J^1 \pm iJ^2)$ show that the operator product algebra of these currents is

$$J^a(z)J^b(0) \sim \frac{\delta^{ab}}{z^2} + i\sqrt{2}\epsilon^{abc}\frac{J^c(0)}{z} + \ldots$$

(d) [Bonus tedium] Defining modes as usual for a dimension 1 operator,

$$J^a(z) = \sum_{n\in\mathbb{Z}} J^a_n z^{-n-1}$$

show that

$$[J^a_m, J^b_n] = i\sqrt{2}\epsilon^{abc}J^c_{m+n} + mk\delta^{ab}\delta_{m+n}$$

with $k = 1$, which is an algebra called Affine $SU(2)$ at level $k = 1$. Note that the $m = 0$ modes satisfy the ordinary $SU(2)$ lie algebra.

(e) Think about how the results of (a)-(d) verify the claim that the spectrum of the compactified theory at this special radius really has non-abelian gauge symmetry, with the extra gauge bosons made from wound strings. Specifically, construct the physical vertex operators for the three $SU(2)_L$ gauge bosons by tensoring J^a with some right-moving operator (remember to level-match) and some factor that allows the resulting state to have (null) momentum in the noncompact dimensions. Remember that an operator $e^{ik_L X_L}e^{ik_R X_R}$ creates a string mode with nonzero winding if $k_L \neq k_R$.