The 2d harmonic oscillator (for the relative coordinate) is of course solved with creation and annihilation operators. The states of definite L arise by acting with

$$(a_x^\dagger + i a_y^\dagger)^{L+p}(a_x^\dagger - i a_y^\dagger)^p$$

on the ground state. Both $L+p$ and p must be ≥ 0. The energy depends on $N = L + 2p$. Thus the degeneracy is 0, with $L = 0$, for $N = 0$, 2, with $L = \pm 1$, for $N = 1$, 3, with $L = -2, 0, 2$, for $N = 3$, etc.

For bosons we must have $L =$ even, for fermions $L =$ odd. If we make a small positive addition to L, the states with $L \geq 0$ will go up in energy, while those with $L < 0$ will go down. Thus we get the spectral flow (topologically, by this argument, but also analytically):

To do this for a particle, we introduce the gauge field a and the couplings

$$\Delta\mathcal{L} = q\vec{v} \cdot a + \kappa\epsilon^{\alpha\beta\gamma} a_\alpha f_{\beta\gamma}$$

The equation of motion for a then reads

$$
\begin{aligned}
\Delta \mathrm{L} &= \int dt r F_\phi \\
&= \int r q E_\phi \\
&= \frac{q}{2\pi} \int dt \frac{\partial \Phi}{\partial t} \\
&= \frac{q\Phi}{2\pi}
\end{aligned}
$$

Aharonov and Bohm analyzed the scattering of a charged particle off a flux tube in their original, classic paper.

The effect of the flux tube, as we've seen, is to shift the angular momentum spectrum. The effect of that on the Bessel functions that appear in a partial wave analysis of the incoming plane wave is different for $L \geq 0$ and $L < 0$. Indeed, the relevant (non-singular) Bessel functions are the J_L for $L \geq 0$ and the J_{-L} for $L < 0$, so increasing L increases the index in one case, decreases it in the other!

That heuristic suggests the form of the scattered wave. We have an overall kinematic factor $\frac{e^{ikr}}{\sqrt{2\pi kr}}$. We also have a uniform angular factor $e^{i\alpha\phi}$ (where $\alpha \equiv \frac{q\Phi}{2\pi}$ is the angular momentum shift) and finally the phase-shifted Bessel asymptotics

$$\begin{aligned}
\sum_{n=0}^{\infty} e^{in\phi} e^{i\alpha} + \sum_{n=-1}^{\infty} e^{in\phi} e^{-i\alpha} &= e^{i\alpha} \frac{1}{1 - e^{i\phi}} + e^{-i\alpha} \frac{e^{-i\phi}}{1 - e^{-i\phi}} \\
&= (e^{i\alpha} - e^{-i\alpha}) \frac{1}{1 - e^{i\phi}} \\
&= -e^{-i\phi/2} \frac{\sin\alpha}{\sin(\phi/2)}
\end{aligned}$$

From this we extract the cross-section

$$\frac{d\sigma}{d\phi} = \sin\alpha \frac{1}{2\pi k} \frac{1}{\sin^2(\phi/2)}$$

Making the gauge transformation $a_\mu \to a_\mu + \partial_\mu \Lambda$ and integrating by parts:

$$\int \Theta(S)\kappa\epsilon^{\mu\nu\sigma}\partial_\mu\Lambda f_{\nu\sigma} = -\kappa \int \partial_\mu\Theta(S)\epsilon^{\mu\nu\sigma}\Lambda f_{\nu\sigma}$$