Lecture 3

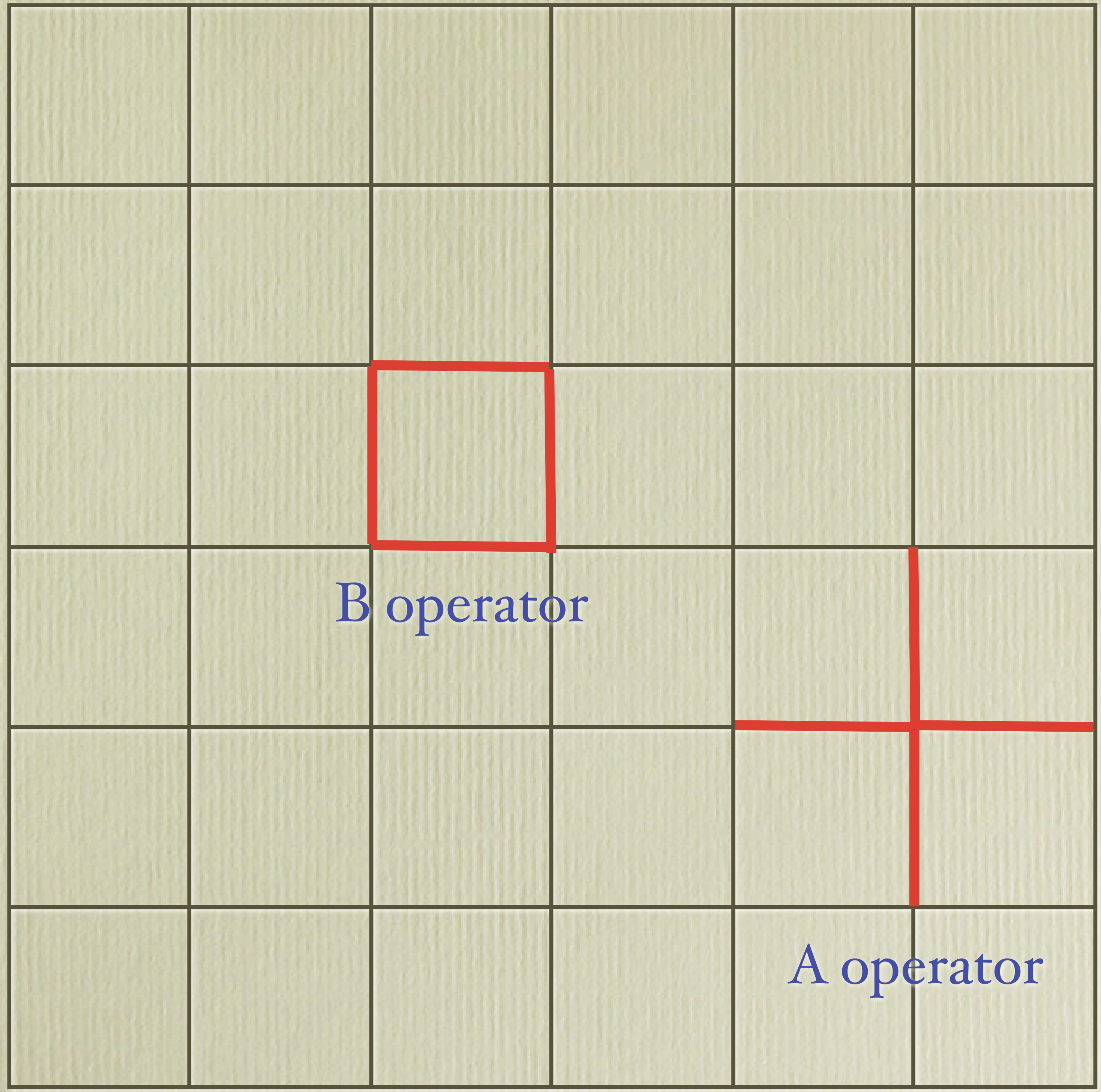
Kitaev's First Model: Degenerate Ground States and Abelian Anyons

Defining the Model

- Consider a kxk square lattice with spins 1/2 ("qubits") on each link - $2k^2$ qubits.
- A complete set of states can be labeled by the eigenvalues of the $\sigma^z{}_1$. If we use |1> for spin up, and |0> for spin down, they become binary arrays.
- Define operators A_s associated with sites and B_p associated with plaquettes:

$$A_j = \prod_{l \,\epsilon\, \mathrm{star}(j)} \sigma_l^x$$

$$B_k = \prod_{l \,\epsilon\, k} \sigma_l^z$$



Lattice with periodic boundary conditions (= torus)

- These operators all commute, because As always meet Bs at an even number of links. Their eignvalues are ±1.

- Define the "protected" states to be those left invariant (eigenvalue +1) by all the As and Bs. These define the allowed words of a simple code with remarkable properties, as we'll see.

- The protected states are also precisely the ground states of the **gapped** Hamiltonian

$$H = -\sum_{\text{sites } s} A_s - \sum_{\text{plaquettes } p} B_p$$

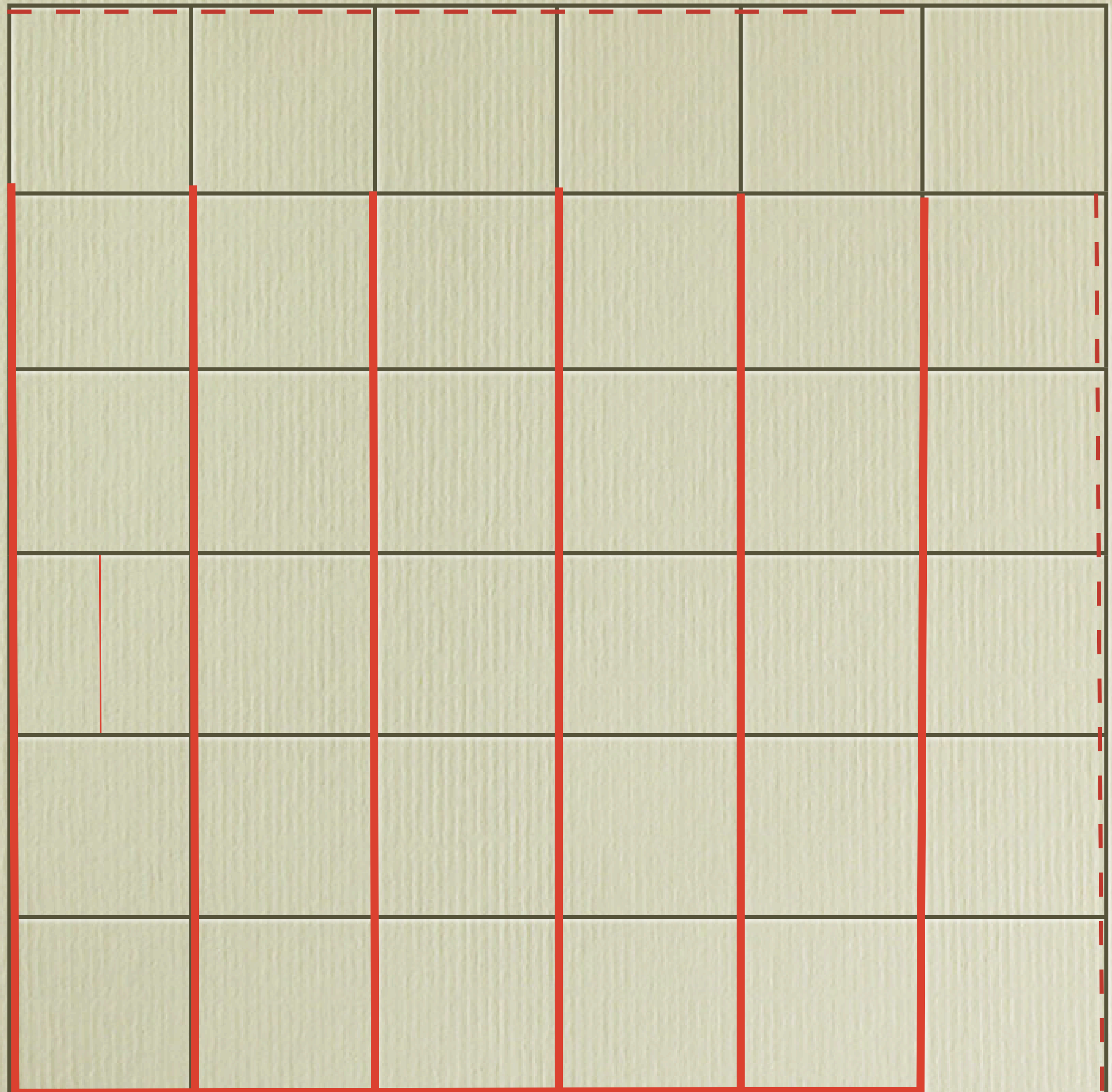
Finding the Protected States

- The B = 1 equations say that the number of down (or up) spins on the links of any plaquette must be even. That condition is of course obeyed by many spin configurations.
- The A = 1 equations will set the coefficients of many such configurations equal. To see how powerful those equations are, we use them to boil down the independent coefficients, by mapping to a few canonical spin configurations, whose coefficients will therefore determine the others.

- A maximal tree is a set of links that contains no loops, but that you can't add to without creating a loop.

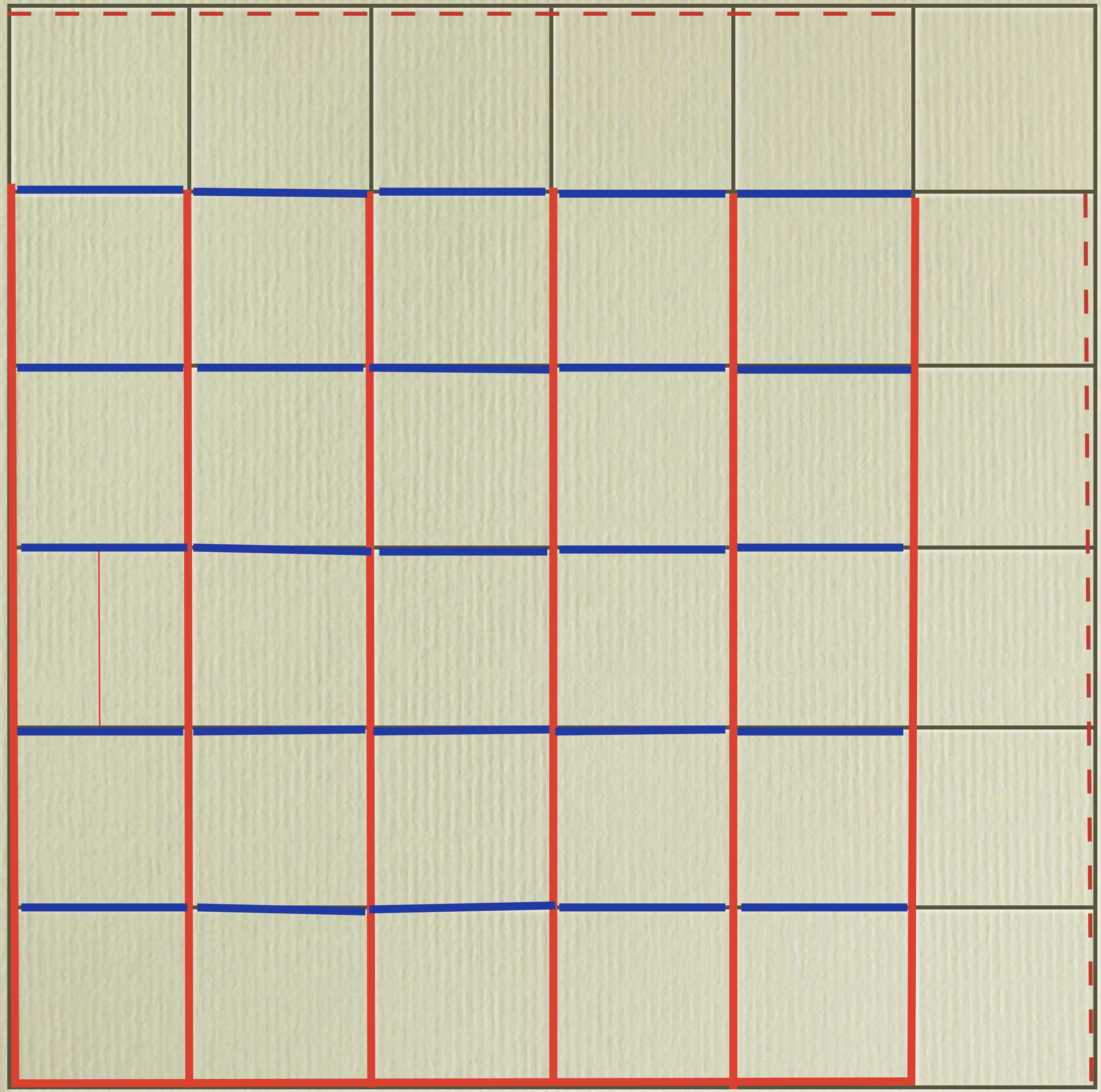
- By applying A operators to a state for which all the $B_k = 1$, you can set all the spins on the links of a maximal tree = $|0\rangle$ (i.e., down). This is similar to gauge fixing in a gauge theory.

- In the following maximal tree, you can run along the top row, then act with As "from the bottom up" one column at a time setting spins = $|0\rangle$. None of these actions interferes with the previous ones.



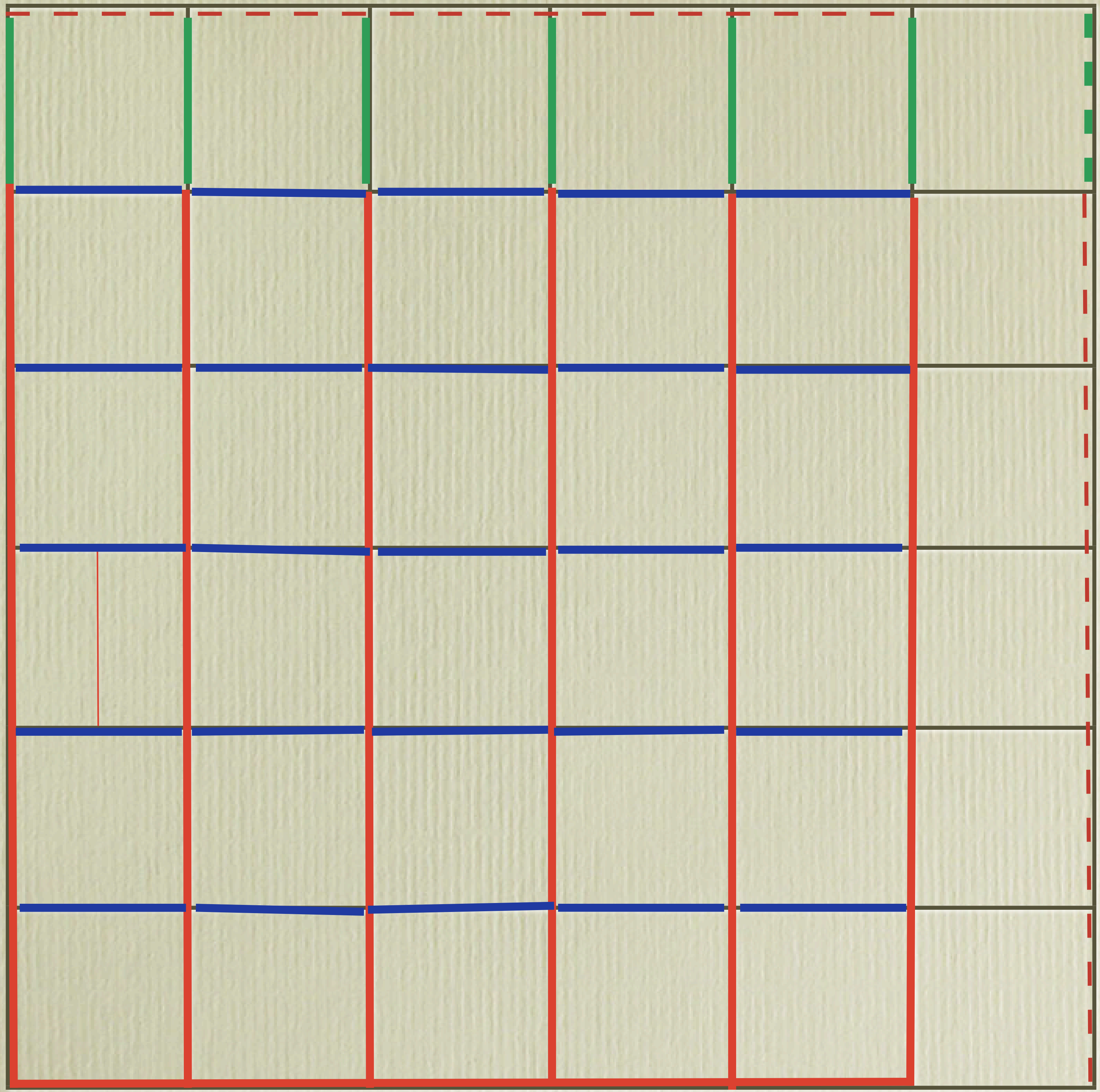
A maximal tree; dotted line is periodic reflection.

- The $B = 1$ condition then enforces many additional spins down:



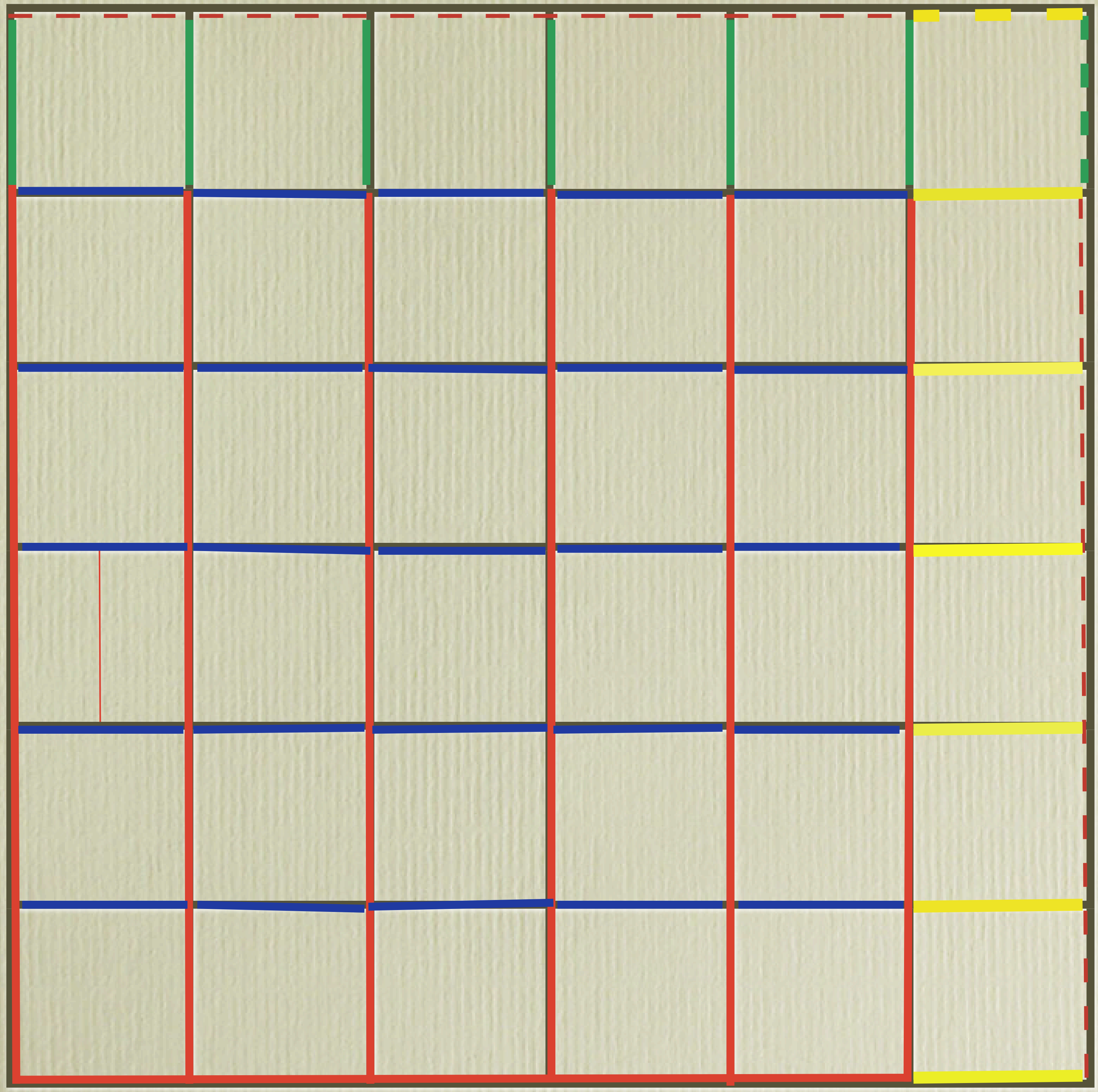
Spins down red links force (B=1) spins down on the blue links too.

- The value of the spin on the northwest link is enforced at other links, again through $B = 1$:



The green links can be up or down, but they must all be equal.

- Similarly, the value at the southeast link is enforced elsewhere ...



The yellow links can be up or down, but they must all be equal.

- The sum (mod 2) of spins along a vertical or horizontal loop cannot be altered by acting with As, so there is no further reduction.

- Thus there are four degenerate ground states. Each consists of a superposition of 1/4 of all possible solutions of the B = 0 equations, each taken with equal weight. The 4 different classes are characterized by the sum of spins along vertical and horizontal loops (mod 2, of course) = (0,0), (0,1), (1,0), or (1,1).

Operator Interpretation

- The existence of 4 code words (alternatively, degenerate ground states) follows heuristically from an operator count. There are k^2 A conditions, k^2 B conditions, and two global relations:

$$\prod_{\text{all sites } j} A_j = 1$$

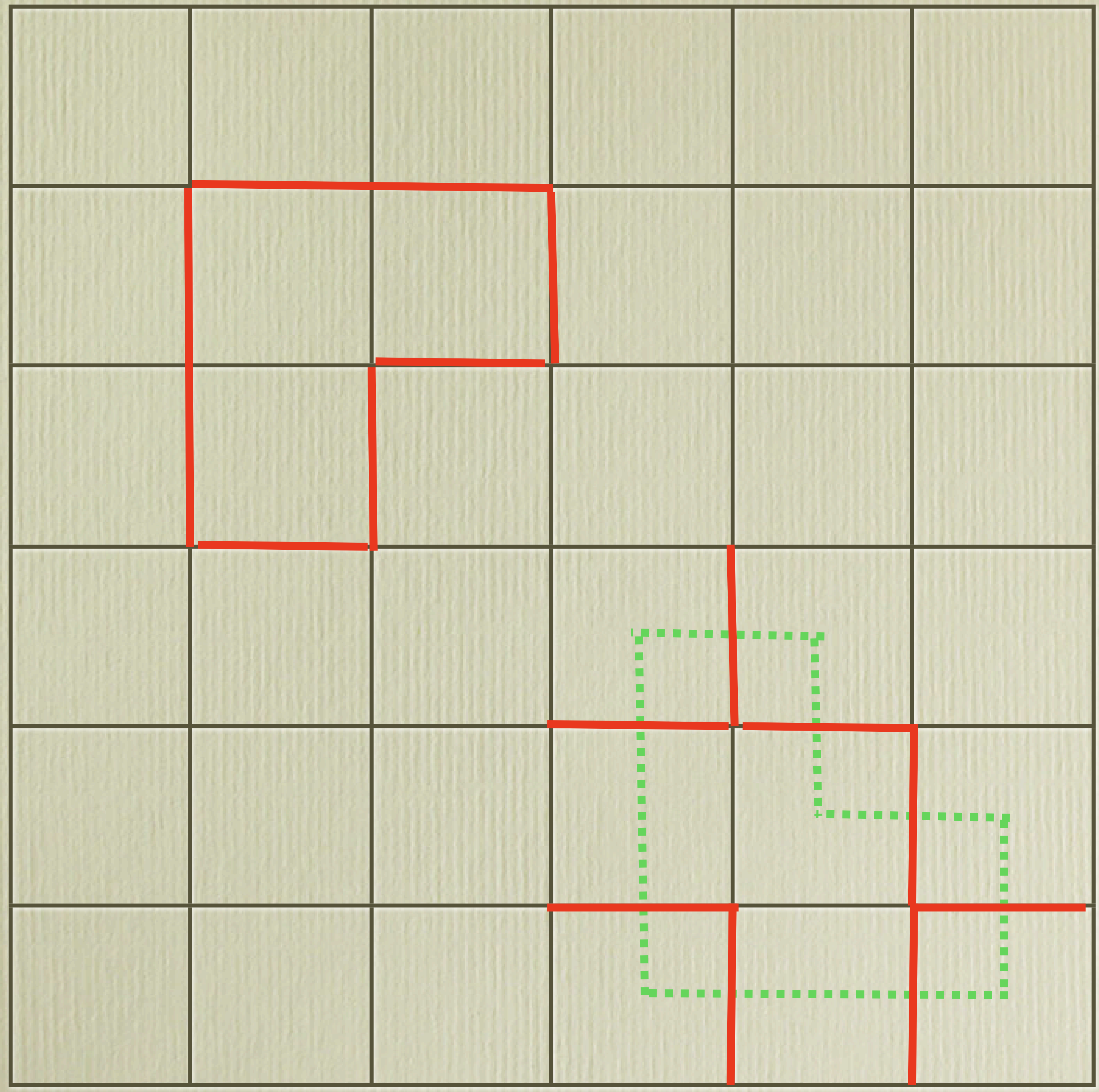
$$\prod_{\text{all plaquettes } p} B_p = 1$$

- Thus there are $2k^2 - 2$ constraints on $2k^2$ spins, leaving 2 spins free. Two free spins span a 4-state Hilbert space.

- A more profound view is based on finding the operators that commute with the As and Bs (a.k.a. the centralizer). There are two classes:

- Z-type operators, based on taking products of $\sigma^z{}_l$ around closed loops.

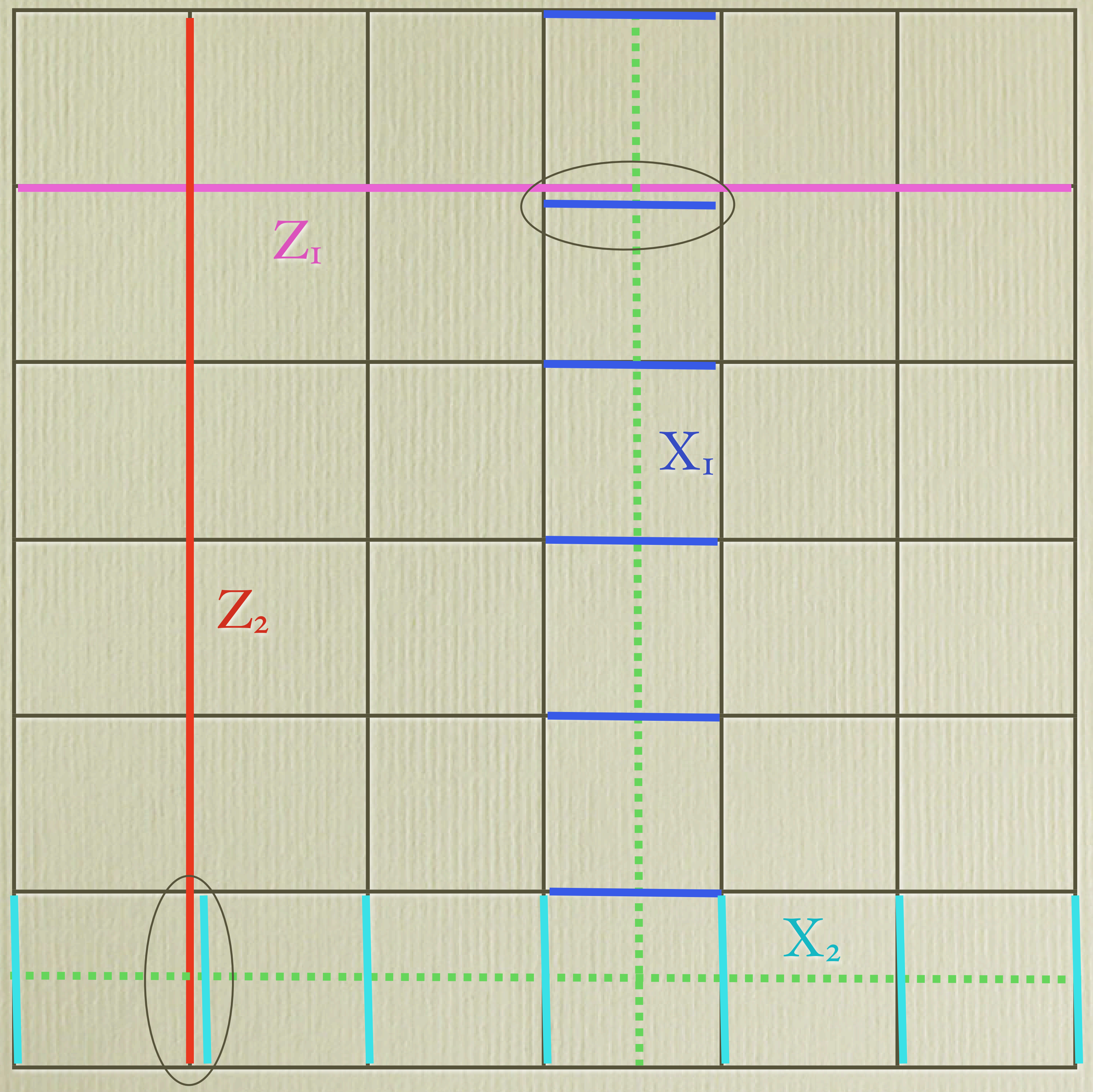
- X-type operators, based on taking products of $\sigma^x{}_l$ around links intersected by closed loops in the dual lattice. A picture is worth ~1000 words here:



- Z operators for contractible loops are products of Bs. (Just plaster the inside with plaquettes.)

- X operators for contractible loops are products of As. (Plaster the inside with plaquettes of the dual lattice, and use sites at the centers of those plaquettes.)

- There are 2 non-contractible loops, and corresponding Z and X operators:


Z1
X1
Z2
X2

- X_1 and Z_1, and separately X_2 and Z_2, satisfy the commutation relations of σ^x and σ^z. Thus there is an SU(2)xSU(2) algebra realized on our four states.

- We have constructed 2 **protected** qubits.

Information Storage

- What kind of foul-up would cause us to mistake one codeword - that is, one of our protected qubits - for another?
- Consider the generic Error-inducing operator

$$
\begin{aligned}
E &= \prod_j (\sigma_j^x)^{\alpha_j} \prod_k (\sigma_k^z)^{\beta_k} \\
\alpha_j &= 0 \text{ or } 1 \\
\beta_k &= 0 \text{ or } 1
\end{aligned}
$$

- It will take us to a new codeword only if E commutes with all the As and Bs.

- But if E is made from As and Bs, it doesn't change the codeword, and there is no mistake.

- E can only cause mistaken identity if it contains a non-contractible loop (or dual loop). But that requires $\geq$ k errors!

- We can do local checks with the As and Bs, that can reliably detect up to k-1 errors, and can therefore reliably correct up to [k-1/2] errors.

- We can do the correction by applying the projector $\Pi\ (1+A)\ \Pi\ (1+B)$.

- Since the codewords are ground states, errors are excitations, and one could imagine that cooling could eliminate them physically!

- If we consider local perturbations, the implication is that overlap between protected codewords be induced only at high orders, ~k, in perturbation theory. Thus it is exponentially small in the size of the lattice.

Electric and Magnetic Excitations

- Since the As and Bs commute with the Hamiltonian, and each other, we can diagonalize the Hamiltonian using their eigenstates.

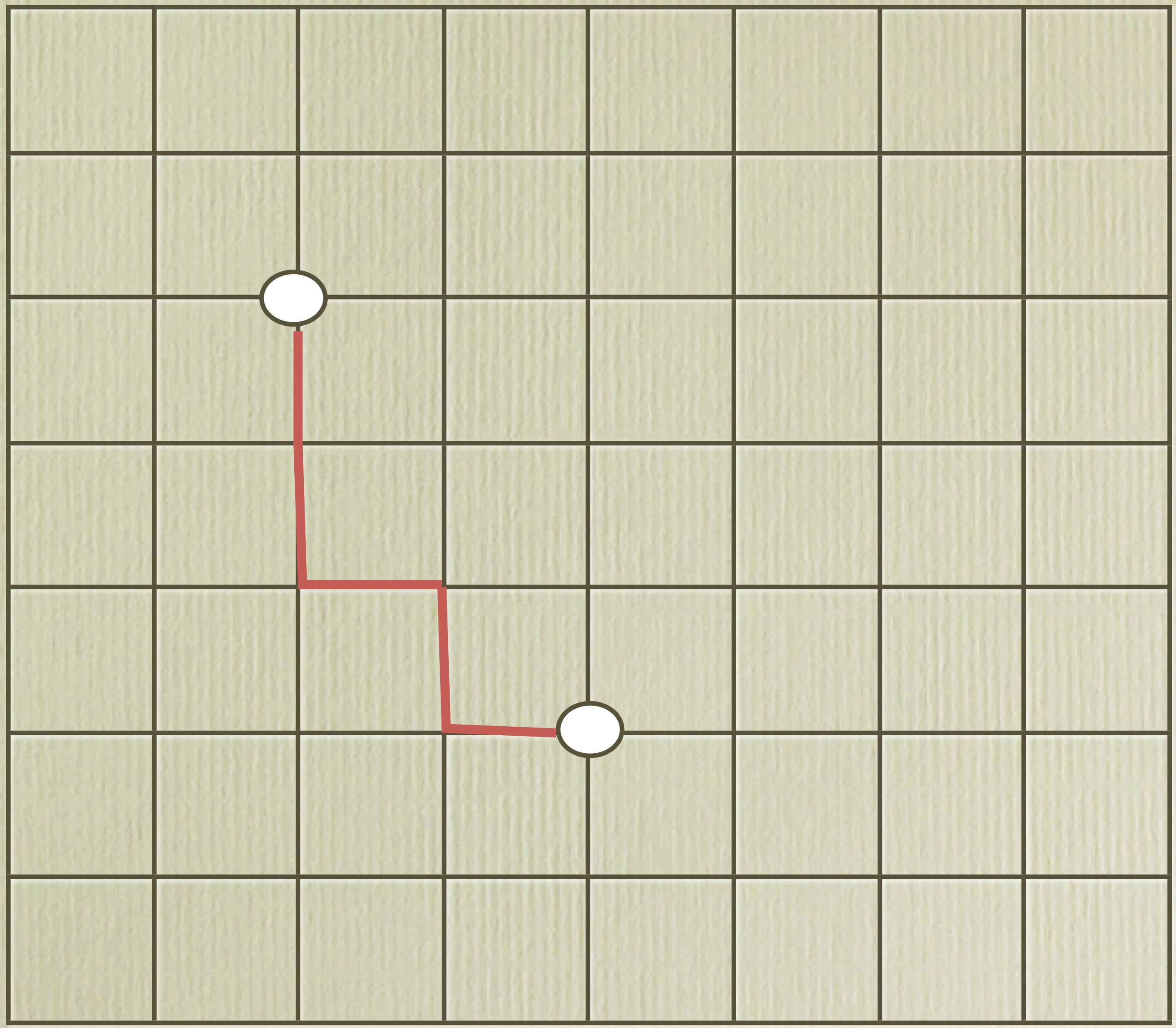
- The excitations of our model Hamiltonian have energy measured by the number of stars and plaquettes they bring from 1 to -1.

- Due to the global constraints on the As and Bs, it is impossible to frustrate just one star or just one plaquette.   The minimum is two.

- Electric pairs can be created by **open** Z-type operators.

- No plaquettes are excited. The excited stars occur for sites at the end of the string. We say electric particles are at these sites.

- Since B operators move the string around, the connecting string can be jiggled around without changing the state. Its topology does matter, however.   (Strings that wrap around cycles change the state, as we've seen with Z_1 and Z_2.)

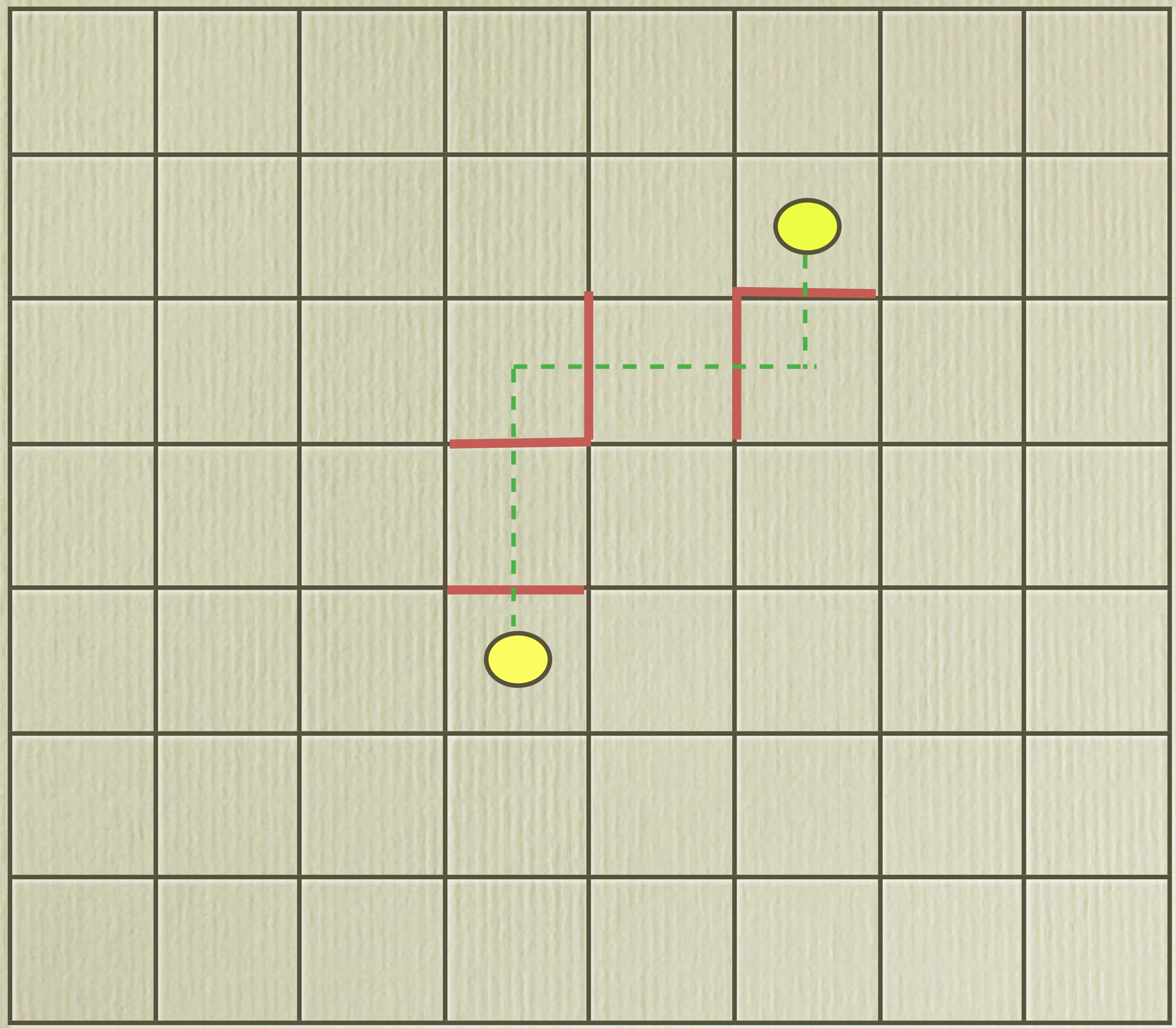


Electric pair with string

- Magnetic pairs can be created by **open** X-type operators.

- No stars are excited. The excited plaquettes occur around the (dual) sites at the end of the string. We say magnetic particles are at these sites.

- Since A operators move the string around, the connecting string can be jiggled around without changing the state. Its topology does matter, however. (Strings that wrap around cycles change the state, as we've seen with X_1 and X_2.)



Magnetic pair with string

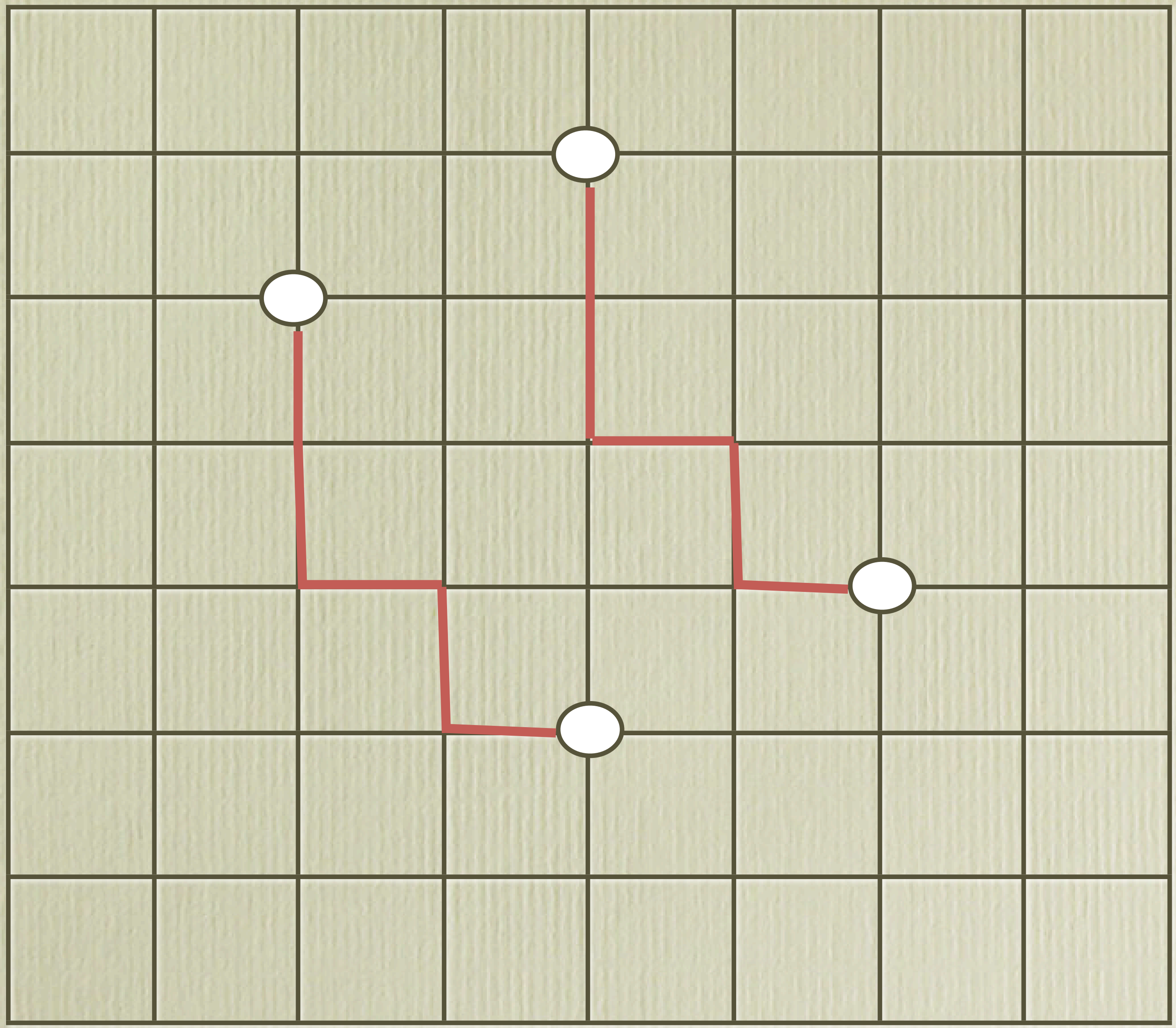
Mutual Anyonic Statistics

- Many-particle eigenstates, containing both electric and magnetic particles, can be constructed by plunking down multiple (non-crossing) open strings. The energies just add. Electric charges interact only by annihilating, as do magnetic charges.

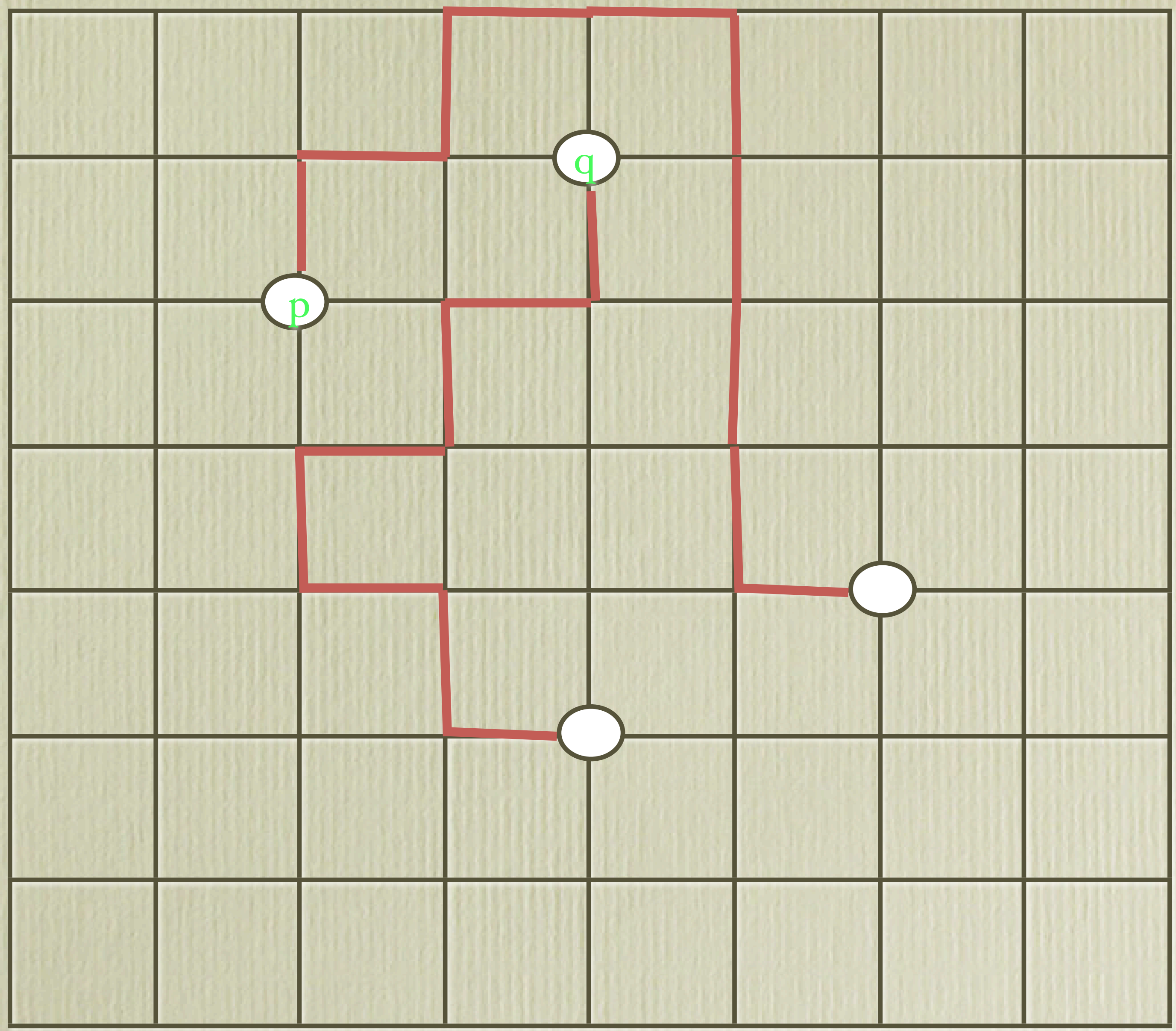
- The topology of the string network matters, however. Different topologies may define different states, or the same state with a different phase.

- Electric strings can be pulled through one another (since all the σ^z commute) as can magnetic strings (since all the σ^x commute).

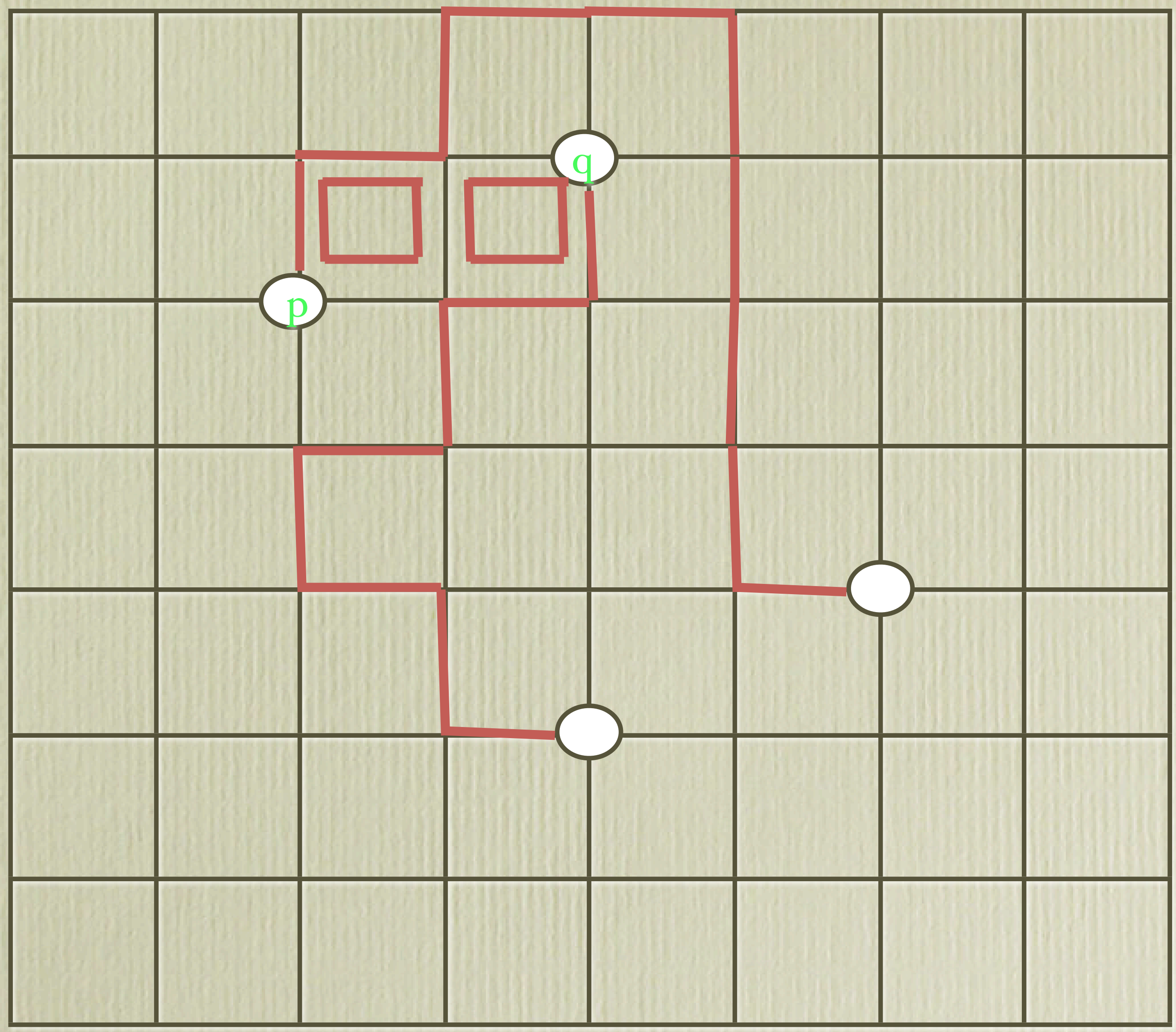
- Interchange of electric - or magnetic - particles gives back the same state. (See following Figures.) Thus, taken separately, they are bosons.



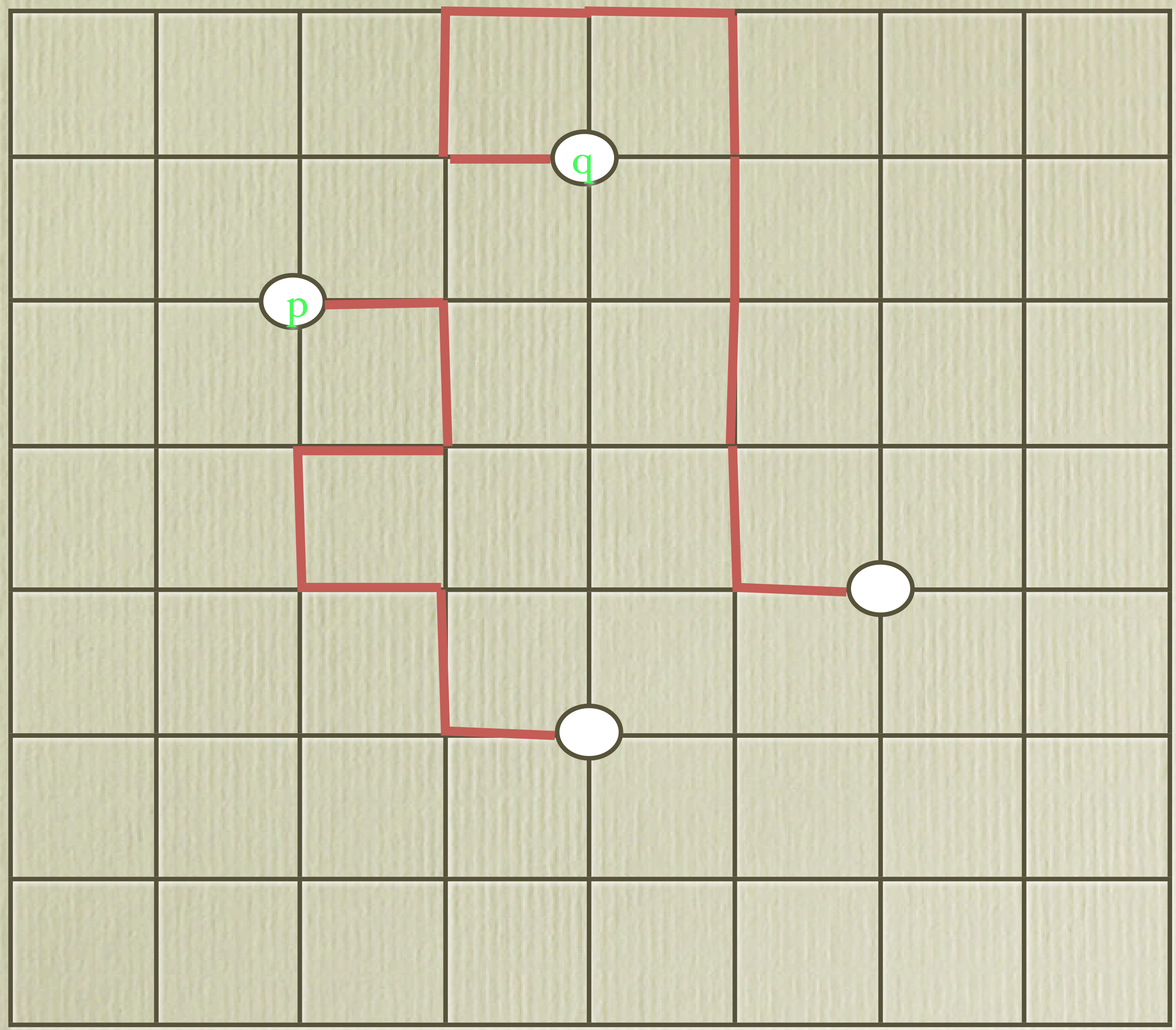
Two electric pairs with strings



Interchange, with p going over q

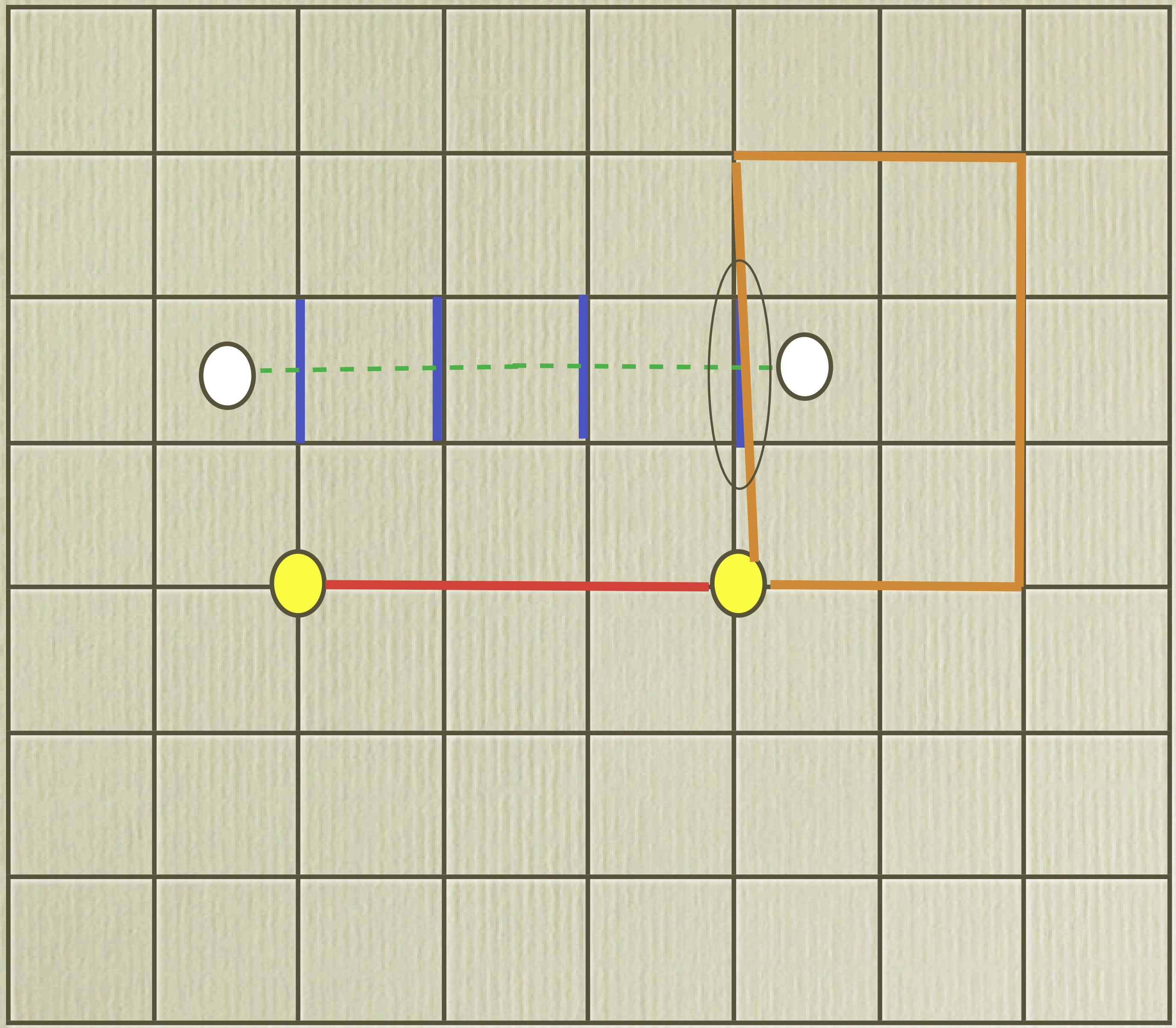


Getting back, with B operators



Reconnecting, with B operators

- But pulling an electric particle around a magnetic particle gives a minus sign, as a σ^z gets pulled through a σ^x:



The extra (orange) loop induces a factor -1

More formally, in operator language:

In the absence of the magnetic pair, our closed-loop Z type operator leaves the (that is, any of the 4!) ground state invariant:

$$Z_{\text{loop}}|0\rangle = |0\rangle$$

But in the presence of the X string there is a nontrivial commutator, and we get a factor -1:

$$Z_{\text{loop}}(X_{\text{string}}|0\rangle) = Z_{\text{loop}}X_{\text{string}}|0\rangle = -X_{\text{string}}Z_{\text{loop}}|0\rangle = -(X_{\text{string}}|0\rangle)$$

- This behavior is unlike conventional quantum particle behavior. The electric and magnetic particles have *mutual* anyon statistics.

- It is a subtle long-range quantum interaction, in a system with an energy gap.

Anyons and Ground State Degeneracy

- There is a close relationship between the existence of anyons and ground state degeneracy.
- If we produce a virtual electric pair, bring it around a closed loop, and annihilate, it leaves behind our X_1 operator!

Scholium

- With the schematic - Σ (A + B) Hamiltonian, the electric and magnetic particles at definite positions are exact eigenstates. There is no tendency to "hop". With a more general Hamiltonian, of course, that wouldn't necessarily be the case.

- Exercise, I think: Go from simple “sign” operators to “phase” operators, and get true anyons.
- Exercise: Work out statistics of dyons.
- Project: Make models in this spirit for higher dimensional situations and objects.