Postscript to Lecture 3

Confinement

- This model also supports a simple and instructive model of confinement.

- The ground state(s) can be constructed in another way, by projecting onto the +1 eigenstates of the As and Bs directly:

$$|\text{vacuum}\rangle = \prod \frac{1+A}{\sqrt{2}} \prod \frac{1+B}{2} |0\rangle$$

where $|0\rangle$ is the "all down" state. The second factor is redundant, but serves to remind us that the vacuum contains a magnetic as well as an electric condensate. Also, with this factor present, the operator product projects *any* state that it doesn't annihilate to a ground state. The square root insures proper normalization. In

$$(1+A)^2 = 1 + 2A + A^2 = 2 + 2A$$

the term with A flips spins, and has zero matrix element. (There's a small loophole here, since in principle a whole bunch of As could conspire to undo one another's flips – but that doesn't happen.)

- Of course, we could choose instead to let |0> be the states of all x-components of spins down; then the first factor is redundant.

- Now consider an electric source with a *small* charge q, rather than "unit" charge. We produce a pair by having a modified open Z-type string, in which we have a product of operators $e^{2\pi i q \sigma^z}$ instead of simply σ^z (which corresponds to q = 1/2). Note that now the orientation of the string matters.

- We want to evaluate the energy of a separated charge-anticharge pair. The Hamiltonian has A terms and B terms. The matrix element of the B terms is unaffected by the presence of the pair and its string.

- But the matrix elements of the A terms are affected, since the string does NOT anticommute with the A operators; indeed $e^{2\pi i q \sigma^z} A = A\, e^{-2\pi i q \sigma^z}$ if A contains the link where σ^z lives.

- For an A term that intersects a given bit of the string we have:

$$
\begin{aligned}
\langle 0| \prod \frac{1+A}{\sqrt{2}} \prod e^{-2\pi i q \sigma^z} A \prod e^{2\pi i q \sigma^z} \prod \frac{1+A}{\sqrt{2}} |0\rangle &= \langle 0| \prod \frac{1+A}{\sqrt{2}} A \prod e^{4\pi i q \sigma^z} \prod \frac{1+A}{\sqrt{2}} |0\rangle \\
&= e^{-4\pi i q} \langle 0| \prod \frac{1+A}{\sqrt{2}} A \prod \frac{1+e^{8\pi i q} A}{\sqrt{2}} |0\rangle \\
&= e^{-4\pi i q} \langle 0| \prod \frac{1+A}{\sqrt{2}} \prod \frac{1+e^{8\pi i q} A}{\sqrt{2}} |0\rangle \\
&= \cos 4\pi q
\end{aligned}
$$

- Each string bit gives us two such factors, since there are two A-operators that it hits; likewise, there A-operator that intersects the string intersects it twice. So we get a contribution of $(\cos 4\pi q)^2$, instead of -1, for each of 2L terms in H, where L is the length of the string.

- Thus the energy of the string is proportional to its length, and the charges are confined.

- An exactly similar argument works for magnetic charge.

- We can say, in a suggestive language, that the (1+A) factors represent a magnetic condensate. Indeed, we associate products of As with X-type operators, which when cut reveal magnetic charges.

- Heuristically, then, the “fractional” electric charge frustrates the magnetic condensate.