Lecture 4

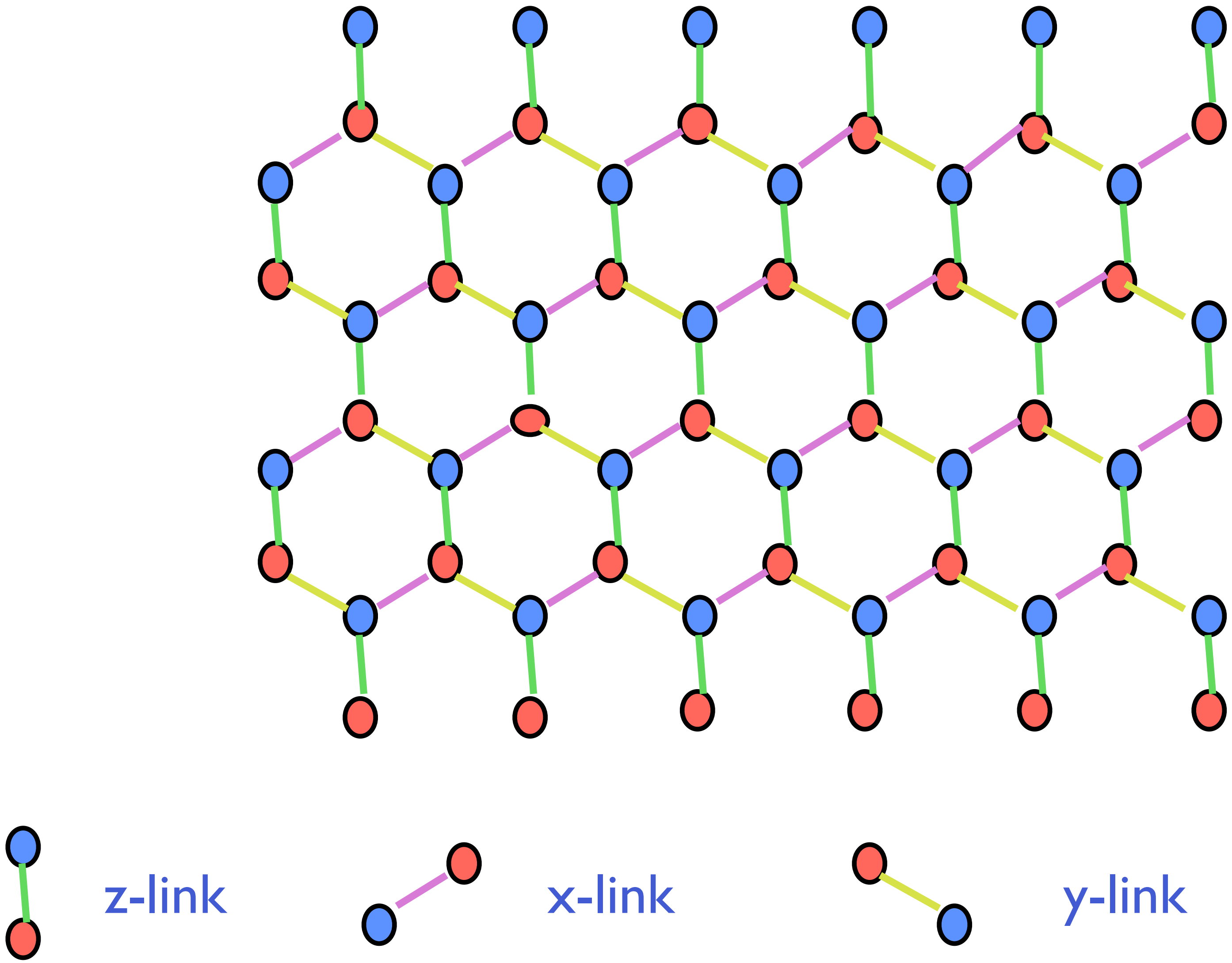
Kitaev's Second Model, Part 1: A Phase With Abelian Anyons

The Model

- We have spins-1/2 at the vertices of a honeycomb lattice.
- The Hamiltonian is

$$H = -J_x \sum_{x \text{ links}} \sigma_j^x \sigma_k^x - J_y \sum_{y \text{ links}} \sigma_j^y \sigma_y^x - J_z \sum_{z \text{ links}} \sigma_j^z \sigma_k^z$$

where the geometry is indicated next:



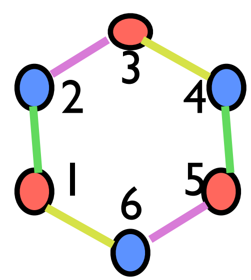
z-link
x-link
y-link

- Note that the honeycomb lattice is bipartite (red vertices have blue neighbors, and vice versa).

- This Hamiltonian, consisting of 2-body interactions, is much more realistic than the 1st Kitaev model.

Plaquette (Hexagon) Operators

- The product of “appropriate spins” around an elementary plaquette, as indicated next, is an integral of the motion.



$$W_p = \sigma_1^x \sigma_2^y \sigma_3^z \sigma_4^x \sigma_5^y \sigma_6^z$$

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- It is very convenient to define the operators K_{ij} on links to be the spin-products that appear in the Hamiltonian, e.g. products of σ^x operators for x-links.

- The W_ps commute with all the K_{ij} - and therefore with the Hamiltonian. Indeed, an external "external" link will impact only one vertex (or none), and will act with the same-direction σ as appears in W_p. And an "internal" link will acquire **two** minus signs on commutation.

$$W_p = \sigma_1^x \sigma_2^y \sigma_3^z \sigma_4^x \sigma_5^y \sigma_6^z$$

- We can also write $W_p = K_{12}K_{23}K_{34}K_{45}K_{56}K_{61}$ (check!). To make it plausible, note that $\sigma^z{}_1$ appears in K_{12} and $\sigma^y{}_1$ appears in K_{61}.
- The W_ps therefore commute with each other, as of course could also be shown directly.

- W_p is Hermitian and $W_p^2 = 1$. So we can partially diagonalize the Hamiltonian by specifying sectors with definite values ±1 of the W_p.

Fermionic Representation of Spins

- From the creation and annihilation operators of fermions we can construct Hermitean operators for their "real and imaginary parts":

$$
\begin{aligned}
c_{2k-1} &= a_k + a_k^\dagger \\
c_{2k} &= \frac{a_k - a_k^\dagger}{i}
\end{aligned}
$$

- These obey the Clifford algebra

$$\{c_j, c_k\} = 2\delta_{jk}$$

- We'll call them Majorana or Majorana fermion operators.

- Given four Majorana operators b^x, b^y, b^z, c, define

$$\begin{aligned} D &\equiv b^x b^y b^z c \\ \tilde{\sigma}^x &\equiv i b^x c \\ \tilde{\sigma}^y &\equiv i b^y c \\ \tilde{\sigma}^z &\equiv i b^z c \end{aligned}$$

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\tilde{\sigma}^x &\equiv i b^x c \\
\tilde{\sigma}^y &\equiv i b^y c \\
\tilde{\sigma}^z &\equiv i b^z c
\end{aligned}
$$

- Each of these is Hermitian and squares to 1.

- D commutes with the tilde-σs.

- Within the subspace with D = 1, the tilde-σ operators obey the standard Pauli σ algebra:

$$
\begin{aligned}
\tilde{\sigma}^x \tilde{\sigma}^y &= i b^x c i b^y c = b^x b^y \\
[\tilde{\sigma}^x, \tilde{\sigma}^y] &= 2 b^x b^y \\
[\tilde{\sigma}^x, \tilde{\sigma}^y]\, D &= 2 b^x b^y b^x b^y b^z c \\
&= -2 b^z c \\
&= 2 i \tilde{\sigma}^z
\end{aligned}
$$

- Our 4 Majorana operators naturally act on a four-dimensional Hilbert space: the Hilbert space of two standard fermions.

- On the two-dimensional subspace with D= 1, they faithfully represent a single spin.

Free Fermions Within the Sectors!

- Now we resume our analysis of H. In terms of the Majorana representation of the spins, the Hamiltonian takes the form (in a condensed but I hope self-explanatory notation):

$$H = \frac{i}{4} \sum_{\text{links}} 2J_{jk} \hat{u}_{jk} c_j c_k$$

where

$$\hat{u}_{jk} \equiv i b_j^{\alpha_{jk}} b_k^{\alpha_{jk}}$$

and α_{jk} is the direction appropriate to the link.

$$\hat{u}_{jk} \equiv i b_j^{\alpha_{jk}} b_k^{\alpha_{jk}}$$

- The u-operators are Hermitian and square to 1.
- They commute with the Hamiltonian, and with each other.
- So we can diagonalize them all, defining various dynamically independent sectors.
- u_{jk} depends, for its sign, on the order of j and k. By convention, when we assign eigenvalues to us we go from even to odd lattice sites (red to blue dots).

$$
\begin{aligned}
D &\equiv b^x b^y b^z c \\
\tilde{\sigma}^x &\equiv i b^x c \\
\tilde{\sigma}^y &\equiv i b^y c \\
\tilde{\sigma}^z &\equiv i b^z c
\end{aligned}
$$

$$
\hat{u}_{jk} \equiv i b_j^{\alpha_{jk}} b_k^{\alpha_{jk}}
$$

- However, the u operators do **not** commute with the Ds.

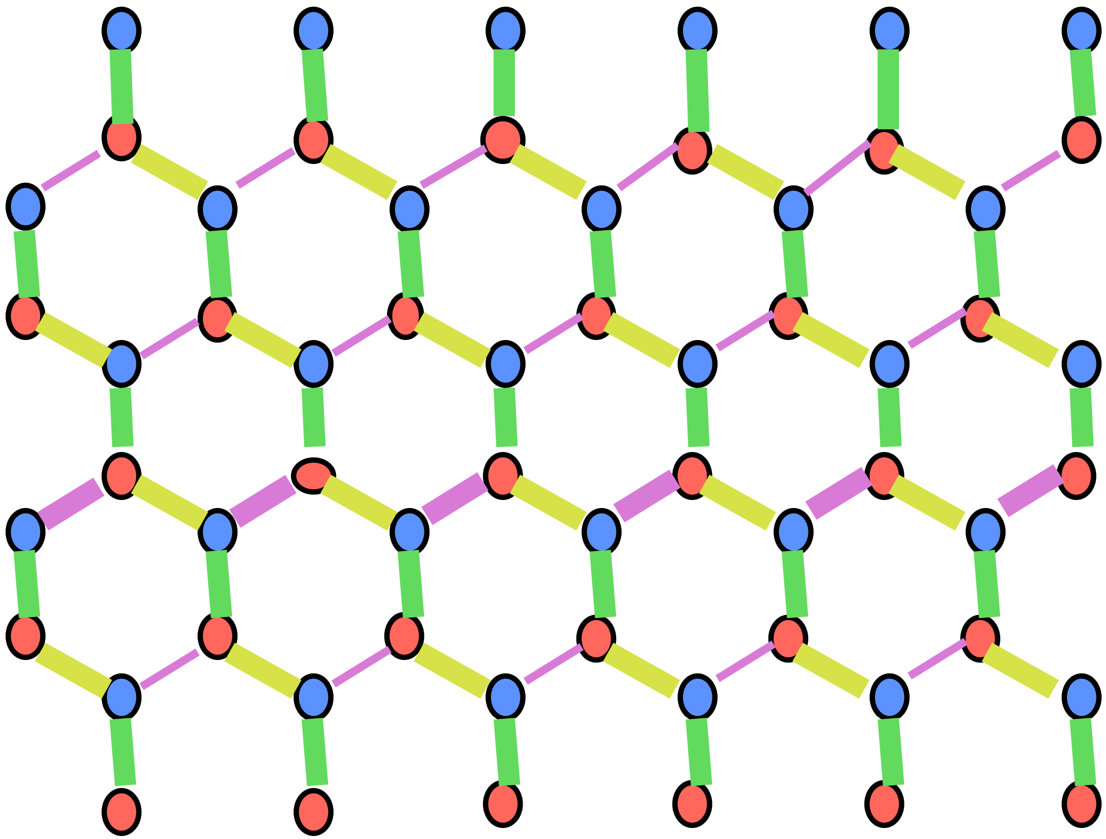
- In order to land within the physical Hilbert space (where the original spins live) we must project with Π(1+D)/2 on all the vertices.

- This step is reminiscent of how we projected with Π(1+A)/2 in the earlier model, to find highly entangled ground states.

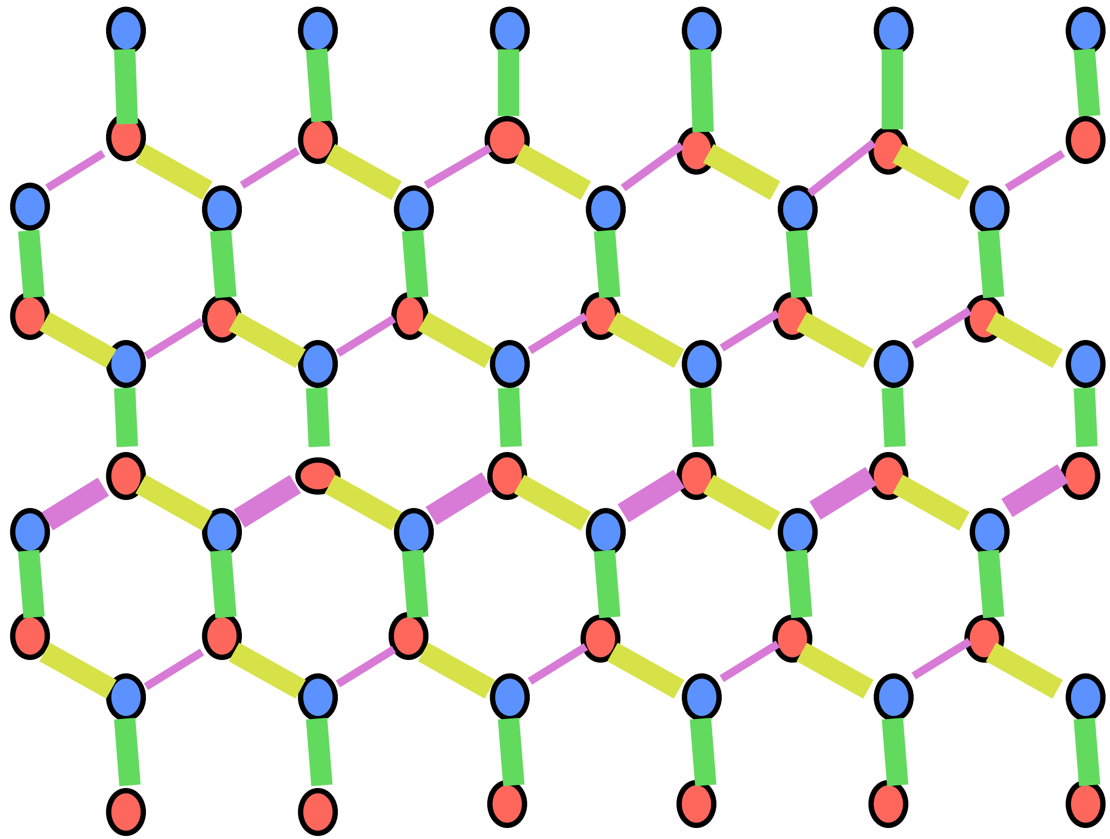
- Here too, in terms of the fermion Fock space, the allowed spin states are highly entangled.
- Q: What do the different u-sectors that get entangled have in common? A: They support the same eigenvalues of the plaquette operators W_p.*

- *This holds strictly for a plane; there is a subtlety on a torus, as will soon appear.

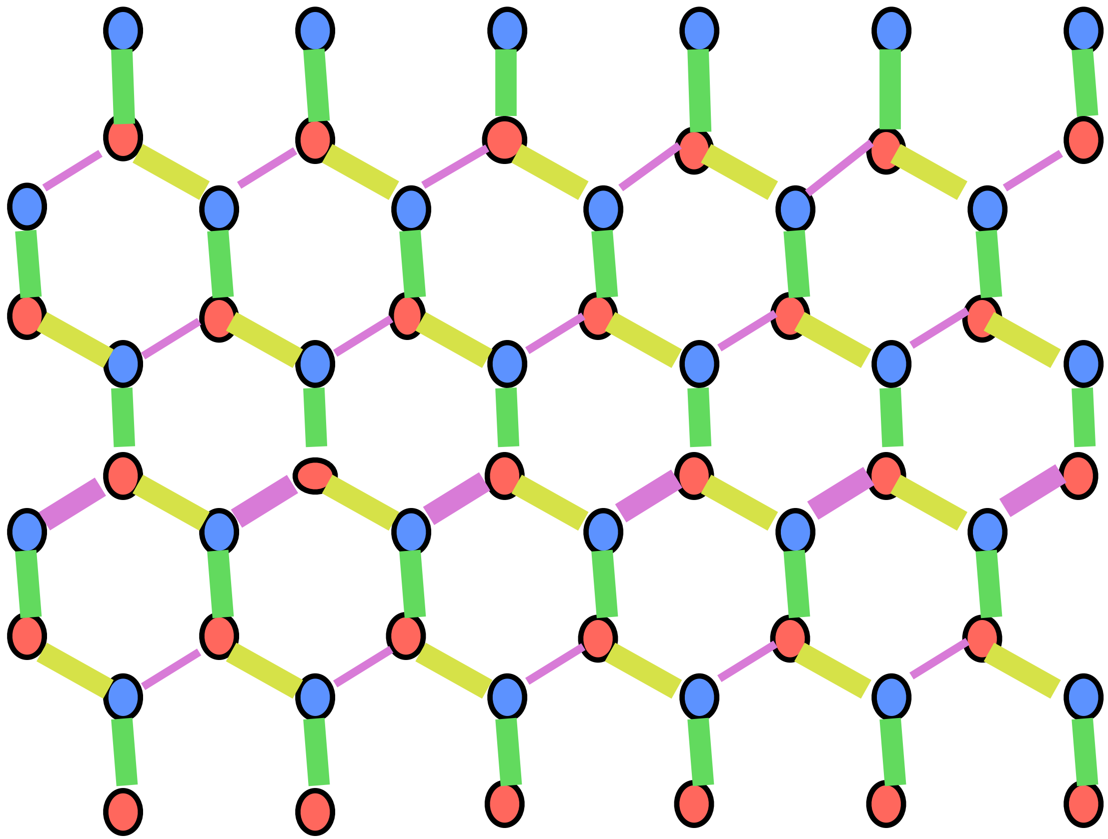
- One shows this by applying D operators (if necessary) to fix the us on a maximal tree = 1:



All the (yellow) y and (green) z links, plus one row of (purple) x links, make a maximal tree.



Using D operators as necessary we can first fix u=1 along the special row, then move from there out through the columns.



The remaining purple us are free. Moving out from the special row, we see that they determine, and are determined by, the plaquette eigenvalues.

- On a torus, we’ll eventually “bite our tail”. This leads to complications reminiscent of what we encountered in the first Kitaev model on a torus (Exercise!).

- Within each sector with definite values of the us, the Hamiltonian becomes a free (i.e., quadratic) fermion Hamiltonian! The hopping coefficients depend on which sector we're in.

$$H = \frac{i}{4}\sum_{\text{links}} 2J_{jk}\hat{u}_{jk}c_jc_k \rightarrow \frac{i}{4}\sum_{j,k} A_{jk}c_jc_k$$

Here A is a big real (and real big) antisymmetric matrix.

Ground State and Spectrum: General Principles

- We now have a set of normal mode problems to solve.
- According to general principles of linear algebra, we can find a real orthogonal transformation Q to put A in the form

$$A = Q \begin{pmatrix} 0 & \varepsilon_1 & & & \\ -\varepsilon_1 & 0 & & & \\ & & \ddots & & \\ & & & 0 & \varepsilon_m \\ & & & -\varepsilon_m & 0 \end{pmatrix} Q^T$$

$$A = Q \begin{pmatrix} 0 & \varepsilon_1 & & & \\ -\varepsilon_1 & 0 & & & \\ & & \ddots & & \\ & & & 0 & \varepsilon_m \\ & & & -\varepsilon_m & 0 \end{pmatrix} Q^T$$

- Here, the energies $\epsilon \geq 0$. Then with

$$(b'_1, b''_1, \cdots, b'_m, b''_m) \equiv (c_1, c_2, \cdots, c_{2m-1}, c_{2m})\, Q$$

we have

$$H = \frac{i}{2} \sum_{k=1}^{m} \varepsilon_k b'_k b''_k = \sum_{k=1}^{m} \varepsilon_k \left(a_k^\dagger a_k - \frac{1}{2} \right)$$

where

$$a_k^\dagger = \frac{1}{2}(b'_k - i b''_k)$$
$$a_k = \frac{1}{2}(b'_k + i b''_k)$$

- The ground states in the different sectors are the totally unoccupied states.   The energy of each ground state is negative. It arises from the 0-point energy of the fermion modes.

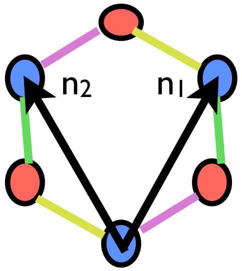
Vortex-Free Sector

- According to a theorem of Lieb, the lowest energy sector is the vortex-free sector. In this sector, we can take all the us = 1. (Note: This includes the even-odd convention for direction. The signs work out: $w_p = \prod u_{jk}$ for each plaquette and the links that belong to it.)
- In the vortex-free sector, we can use Fourier analysis to find the normal modes.

- The straightforward but somewhat unpleasant-to-notate details are left as an exercise. The final result for the energies is

$$\varepsilon(\mathbf{q}) = 2|J_x e^{i(\mathbf{q},\mathbf{n}_1)} + J_y e^{i(\mathbf{q},\mathbf{n}_2)} + J_z|$$

where the n_i are:



$$\mathbf{n_1} = (\frac{1}{2}, \frac{\sqrt{3}}{2})$$

$$\mathbf{n_2} = (-\frac{1}{2}, \frac{\sqrt{3}}{2})$$

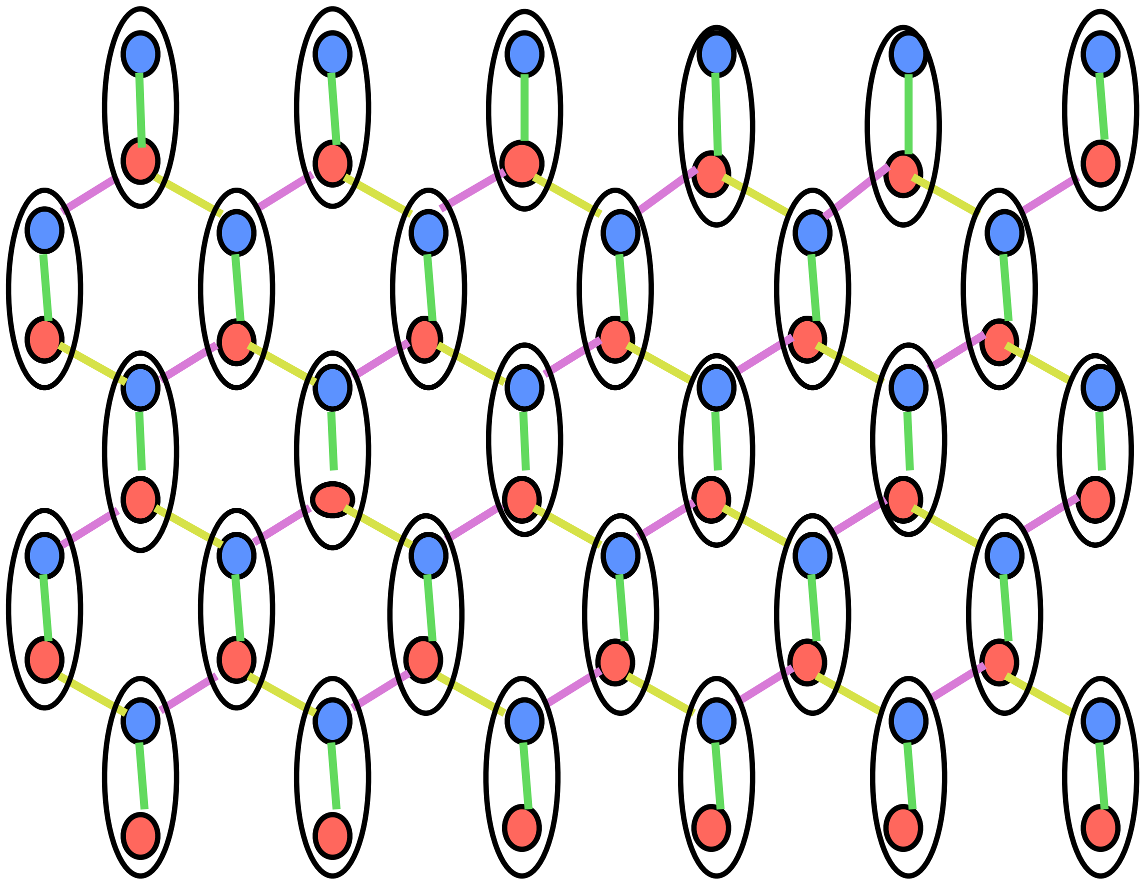
- The qualitative low-energy properties of the system depend on whether there is a gap in the energy spectrum. Thus we must investigate whether there's a solution of $\epsilon(q) = 0$.

- A straightforward analysis (see web page for notes) shows that the necessary and sufficient conditions for there to be a non-trivial solution are the triangle inequalities:

$$
\begin{aligned}
|J_x| &\leq |J_y| + |J_z| \\
|J_y| &\leq |J_z| + |J_x| \\
|J_z| &\leq |J_x| + |J_y|
\end{aligned}
$$

Gapped Phase: Reduction to First Model!

- The long-distance, low-energy behavior of a gapped phase is **semi**trivial: conventional interactions fall off exponentially with distance, but - as we've been obsessing over - there is the possibility of long-range topological interactions, in the form of exotic statistics.
- For definiteness we consider the case $J_z >> |J_x| + |J_y|$.
- If $J_x = J_y = 0$ the Hamiltonian is minimized when spins on z-links are locked together. Thus we have a (trivial) effective spin model for the low-energy states:

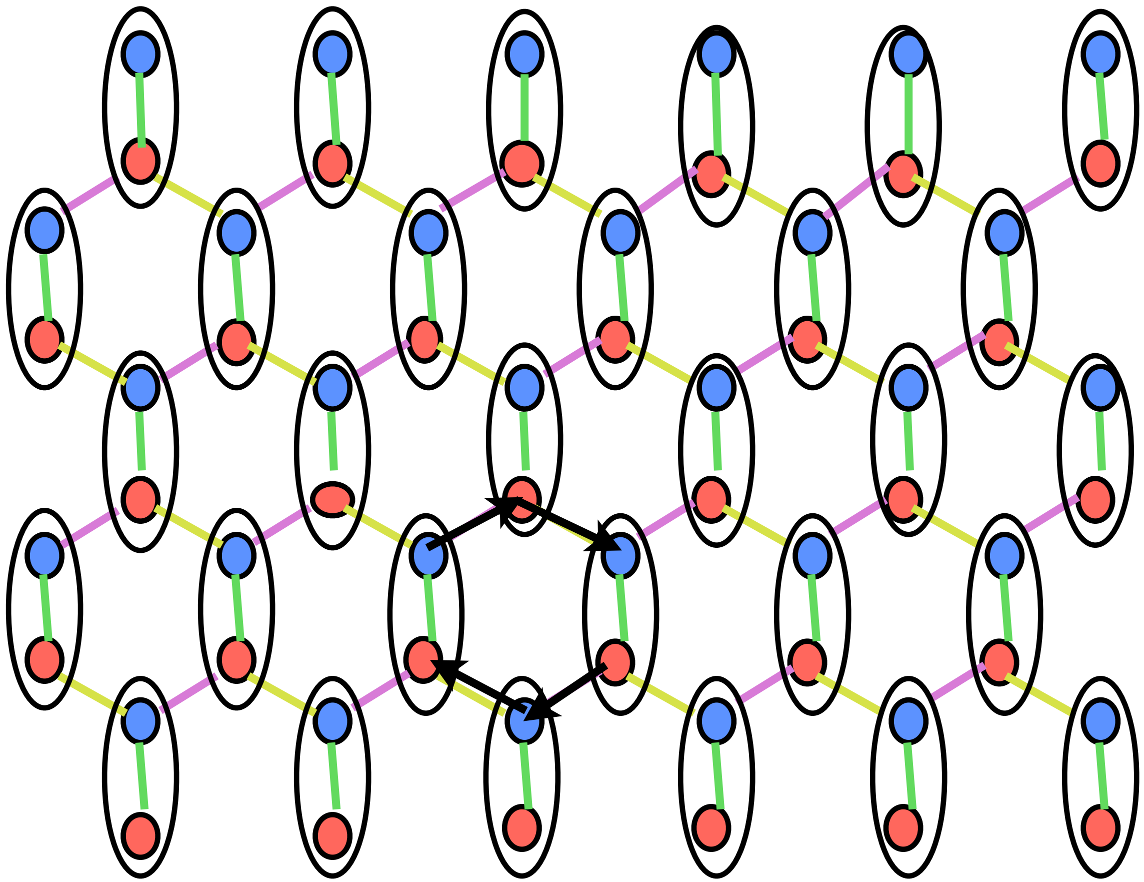


The effective spins live on a square lattice.

- It costs energy of order J_z to break these bonds.

- Now turning on J_x and J_y, we must do degenerate perturbation theory in the states described by the previously uncoupled effective spins.

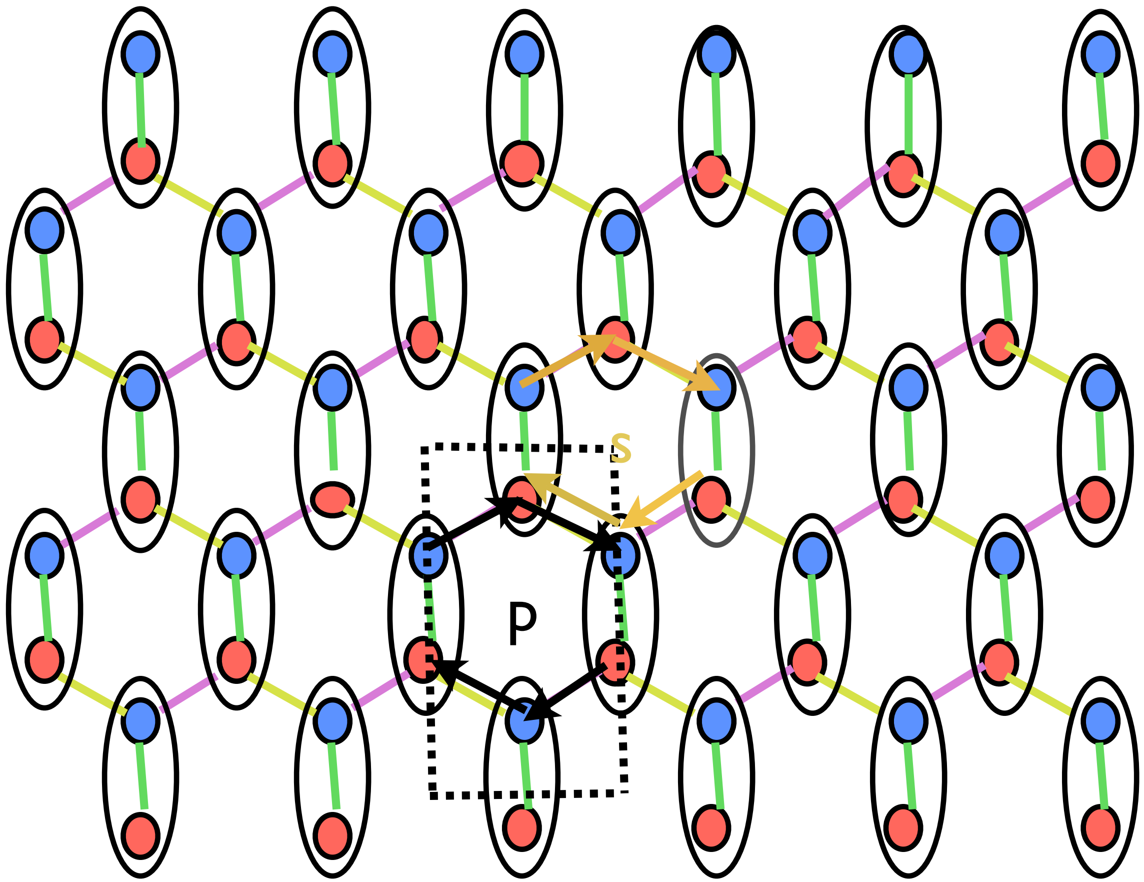
- The first interactions arise in fourth order. It comes from flipping spins in a very particular pattern, since each bonded pair must be hit twice:



Generating interactions among the effective spins.

- A detailed calculation gives, with the geometry shown next,

$$
\begin{aligned}
H_{\text{eff.}} &= -J_{\text{eff.}} \sum_p Q_p \\
J_{\text{eff.}} &= \frac{J_x^2 J_y^2}{16 J_z^3} \\
Q_p &= \sigma^y_{\text{left}(p)} \sigma^y_{\text{right}(p)} \sigma^z_{\text{up}(p)} \sigma^z_{\text{down}(p)}
\end{aligned}
$$



Geometry of effective spin interactions: plaquettes p in the original lattice become plaquettes and stars s in the effective lattice.

- These effective interaction terms have all the essential properties of the A and B operators in the first model. They are Hermitian, mutually commuting, and square to 1.

- Indeed, by a suitable unitary transformation (rotating the spins) we can put the effective Hamiltonian in exactly the earlier form.

Fusion and Braiding

- From our study of the first model, the "superselection" sectors, that cannot be connected to one another by local operators, are vacuum 1, electric excitations e (charges), magnetic excitations m (vortices) and combinations of the two ϵ (dyons). These sectors cannot be reached from one another by local perturbations.
- We have fusion rules for joining the sectors:

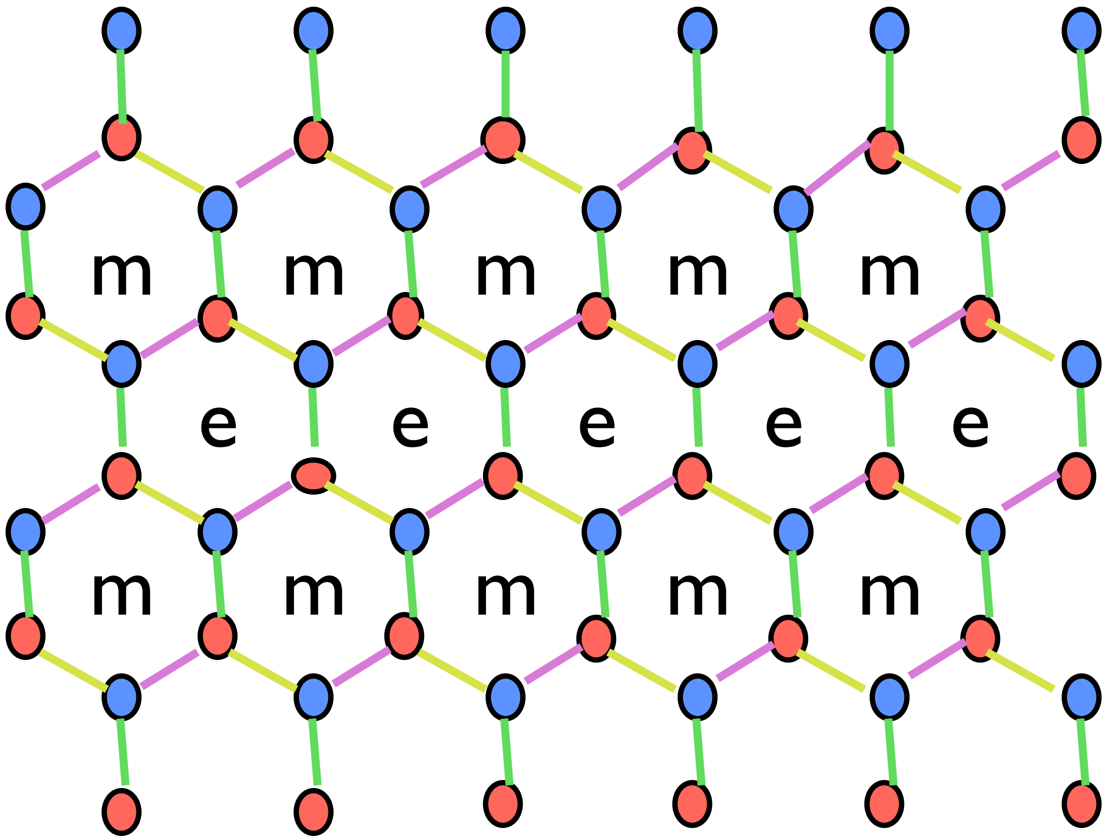
Fusers → ↓	l	e	m	ϵ
l	l	e	m	ϵ
e	e	l	ϵ	m
m	m	ϵ	l	e
ϵ	ϵ	m	e	l

- We showed before that e and m are bosons, with mutual anyon statistics (-1 for braiding).

- At the same time we showed, without perhaps realizing it, that ϵ is a fermion. Indeed, rotating it through 2π brings e around m, hence a - sign; and the Finkelstein-Rubenstein ribbon - or belt! - argument connects spin and statistics.

- (If you did the suggested exercise, you will also have shown this directly.)

- From the point of view of the original spin model, the electric and magnetic excitations are vortices in different parts of the lattice:


m m m m m
e e e e e
m m m m m

- It seems quite remarkable that to transition from one of these vortices to another you must emit a fermion.

- In general this second model, even more that the first, is an amazing example of emergent physics.