Lecture 5

Kitaev's Second Model, Part 2: The Gapless Phase and Magnetic Gapping

Refresher

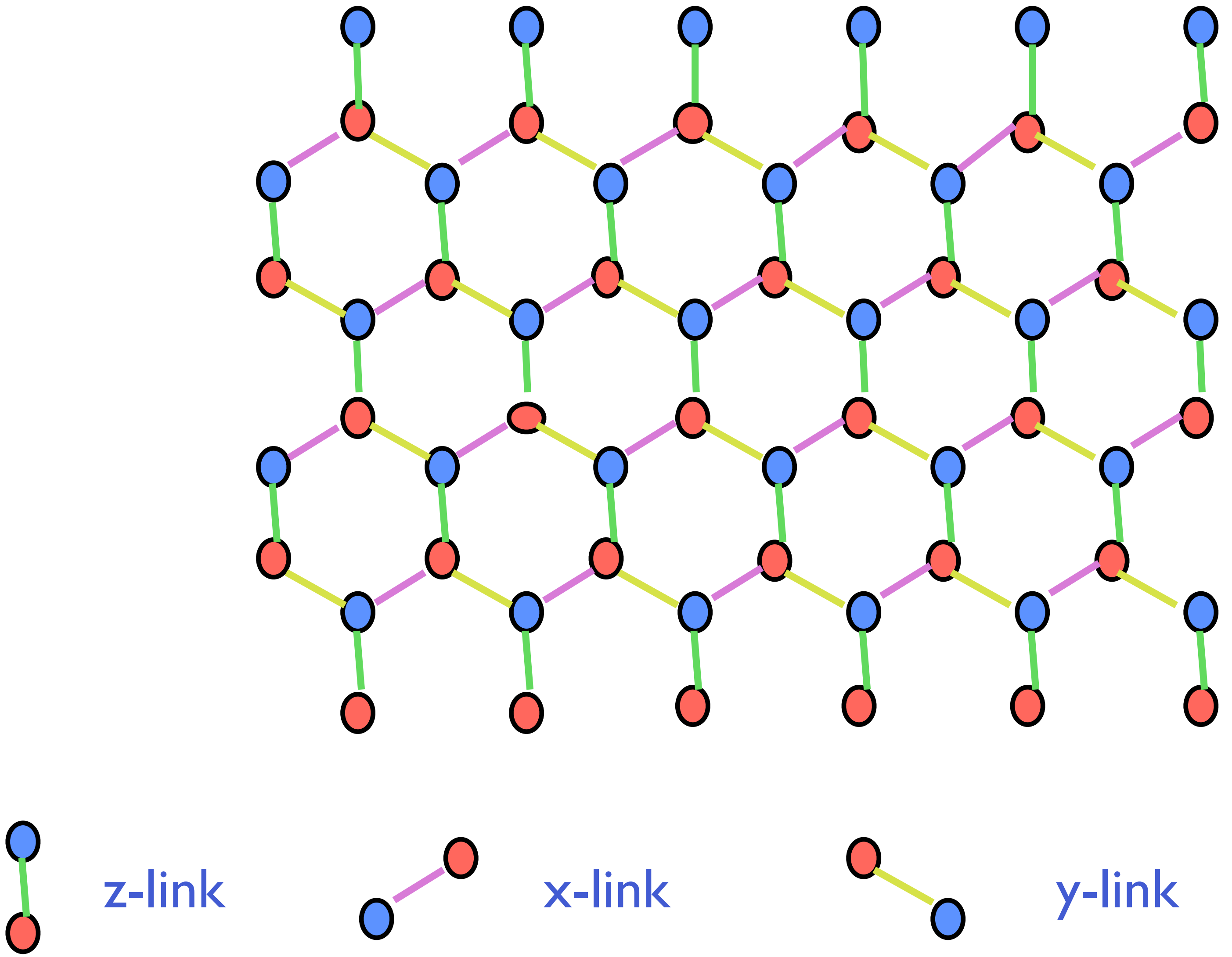
- We are analyzing a model with spins on a honeycomb lattice and the Hamiltonian

$$H = -J_x \sum_{x \text{ links}} \sigma_j^x \sigma_k^x - J_y \sum_{y \text{ links}} \sigma_j^y \sigma_y^x - J_z \sum_{z \text{ links}} \sigma_j^z \sigma_k^z$$

- We found that in each sector - defined by values of the vorticity $W_p = \pm 1$ on each plaquette

$$W_p = \sigma_1^x \sigma_2^y \sigma_3^z \sigma_4^x \sigma_5^y \sigma_6^z$$

the Hamiltonian reduced to a free fermion Hamiltonian.


z-link
x-link
y-link

- The lowest-energy sector is the vortex free sector, with all $W_p = 1$.

- In that sector, the fermion dispersion relation is

$$\varepsilon(\mathbf{q}) = 2|J_x e^{i(\mathbf{q},\mathbf{n}_1)} + J_y e^{i(\mathbf{q},\mathbf{n}_2)} + J_z|$$

- If the Js do not obey the triangle inequalities

$$\begin{aligned} |J_x| &\leq |J_y| + |J_z| \\ |J_y| &\leq |J_z| + |J_x| \\ |J_z| &\leq |J_x| + |J_y| \end{aligned}$$

then there is no solution to $\epsilon(q^*) = 0$: the state is gapped. We analyzed that case last time.

More on the Fermion Spectrum

- Now we turn to the other case. For simplicity we'll suppose $J_x = J_y = J_z = J$.

- Modes for values of q near the zero q* will have very low energy. Phases associated with virtual exchange of those modes will generally dominate and obscure the "statistical" or "topological" phases. For our purposes we will want to gap those modes, but to see how that can work (and for the later analysis) we first need to understand them.

- In more detail, the Fourier analysis (which I did not show) led to a Hamiltonian of the form

$$H = \frac{1}{2} \sum_{\mathbf{q},\text{parities}} i\tilde{A}_{p_1,p_2}(\mathbf{q}) a_{p_1,-\mathbf{q}} a_{p_2,\mathbf{q}}$$

in which the diagonal (same **lattice** parity) terms vanished.

- Thus for each q we have a little matrix

$$\begin{pmatrix} 0 & i\varepsilon(\mathbf{q}) \\ -i\varepsilon(\mathbf{q}) & 0 \end{pmatrix}$$

Implications of Time Reversal

- The peculiar structure of the eigenvalue problem is tied up with time reversal symmetry.
- The original spin model was invariant under the time-reversal symmetry $\sigma \to -\sigma$ for all spins. It can be implemented by (complex conjugation and) $T = i\sigma_2$.
- Expectation values ±1 for the plaquette operators W_p should not spoil this symmetry, since W_p involves an even number of spins (and ±1 is real).

$$\begin{aligned} D &\equiv b^x b^y b^z c \\ \tilde{\sigma}^x &\equiv i b^x c \\ \tilde{\sigma}^y &\equiv i b^y c \\ \tilde{\sigma}^z &\equiv i b^z c \end{aligned}$$

$$\hat{u}_{jk} \equiv i b_j^{\alpha_{jk}} b_k^{\alpha_{jk}}$$

- We can extend the symmetry to the fermion representation of the spins by demanding $Tb_kT^{-1} = b_k$, $TcT^{-1} = c$.

- The sectors with definite values of the us are then not manifestly T invariant. However, we can act with Ds to undo the sign changes within the physical subspace of each sector. (Since the w_ps determine the sector up to D transformations, as we showed with the maximal tree argument). Thus a modified T symmetry, incorporating gauge transformations, remains valid. Locking, again!

- The Hamiltonian

$$H = \frac{i}{4} \sum_{\text{links}} 2 J_{jk} \hat{u}_{jk} c_j c_k$$

is invariant under the modified T, and the us are too. So for the cs we must have:

$$\begin{aligned} T c_j T^{-1} &= c_j \quad \text{j even site} \\ T c_j T^{-1} &= -c_j \quad \text{j odd site} \end{aligned}$$

$$\tilde{a}_{\text{even or odd}}(\mathbf{q}) = \frac{1}{\sqrt{2N}} \sum_{\text{cell translations t}} e^{-i(\mathbf{q},\mathbf{r}_t)} c_{t,\text{even or odd}}$$

$$H = \frac{1}{2} \sum_{\mathbf{q},\text{parities}} i\tilde{A}_{p_1,p_2}(\mathbf{q}) a_{p_1,-\mathbf{q}} a_{p_2,\mathbf{q}}$$

- Thus $Ta_{q,even}T^{-1} = Ta_{-q,even}T^{-1}$ while $Ta_{q,odd}T^{-1} = -Ta_{-q,odd}T^{-1}$.
- Therefore T-symmetry forbids the on-diagonal terms in the Hamiltonian!
- In detail: we get a - sign from the i, and the terms with q and -q get interchanged by complex conjugation in the exponential. But if the parities are the same then those two terms involve the same operators, just in opposite order. So for those diagonal terms we had A(-q) = - A(q) to begin with (also by Hermiticity), while T invariance sets them equal, hence they must vanish.

- This conclusion is very powerful. For one thing, it insures the zero of $\epsilon(q)$ is generic, i.e. it survives small but arbitrary T-invariant perturbations. Indeed, the property of a complex function of two real variables having a zero is robust.

Scholium

- So far our examples of anyon behavior have not involved violation of T symmetry. All we've found is a mutual statistics minus sign on winding - in either direction, of course!

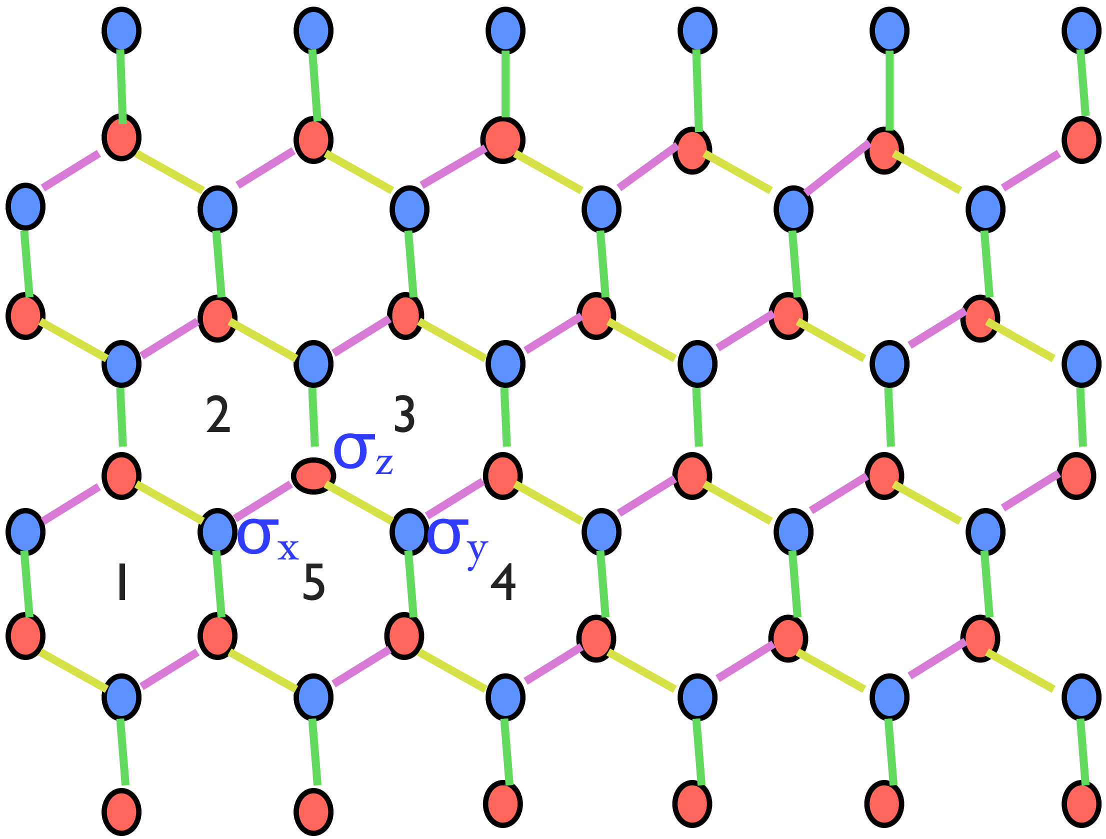
- The true riches of anyon behavior would seem to require either T violation or spectrum doubling. In the latter alternative particles with θ and $-\theta$ statistics get interchaged by T. (As do, for example, magnetic monopoles and antimonoples in T-invariant gauge theories.)

Magnetic Gapping

- We can open the gap with a T-violating external magnetic field, having the effect of adding a perturbation

$$V = -\sum_j (h_x \sigma_j^x + h_y \sigma_j^y + h_z \sigma_j^z)$$

- The gap is of order $h_x h_y h_z/J^2$.
- We are doing degenerate perturbation theory in the vortex-free sector. The first order tern vanishes, because every individual σ anticommutes with two W_ps. The second order term is T-even. There is a third-order "routing" that commutes with all the W_ps, as follows:



The crucial thing to check is that this commutes with all the impacted W_p - which it does.

$$
\begin{aligned}
D &\equiv b^x b^y b^z c \\
\tilde{\sigma}^x &\equiv i b^x c \\
\tilde{\sigma}^y &\equiv i b^y c \\
\tilde{\sigma}^z &\equiv i b^z c
\end{aligned}
$$

$$\hat{u}_{jk} \equiv i b_j^{\alpha_{jk}} b_k^{\alpha_{jk}}$$

- The product is $-iD_{top}u_{side1}u_{side2}c_{side1}c_{side2}$, which reduced to simply $-ic_{side1}c_{side2}$ on the vortex-free physical subspace. It hops directly between even sites, or between odd sites.