Lecture 6

More on the fermion spectrum; background on flows

Road Map of Coming Attractions

- We are in the midst of analyzing Kitaev's second model (involving spins on a honeycomb lattice) in the phase - realized for $J_x = J_y = J_z \equiv J$ - where a small magnetic field produces a gap. Our main goal is to show that the excitations in this phase produces non-abelian anyons.

- The demonstration, and for that matter the result, is rather intricate, so we will do it in steps.

Topology in the Fermion spectrum

$\downarrow$

Flows and Edge Modes from Topology

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Unpaired Majorana Modes on Vortices from Flow

$\downarrow$

Nonabelian Statistics from Unpaired Majorana Modes

Review (with more details) of the Lifted Spectrum

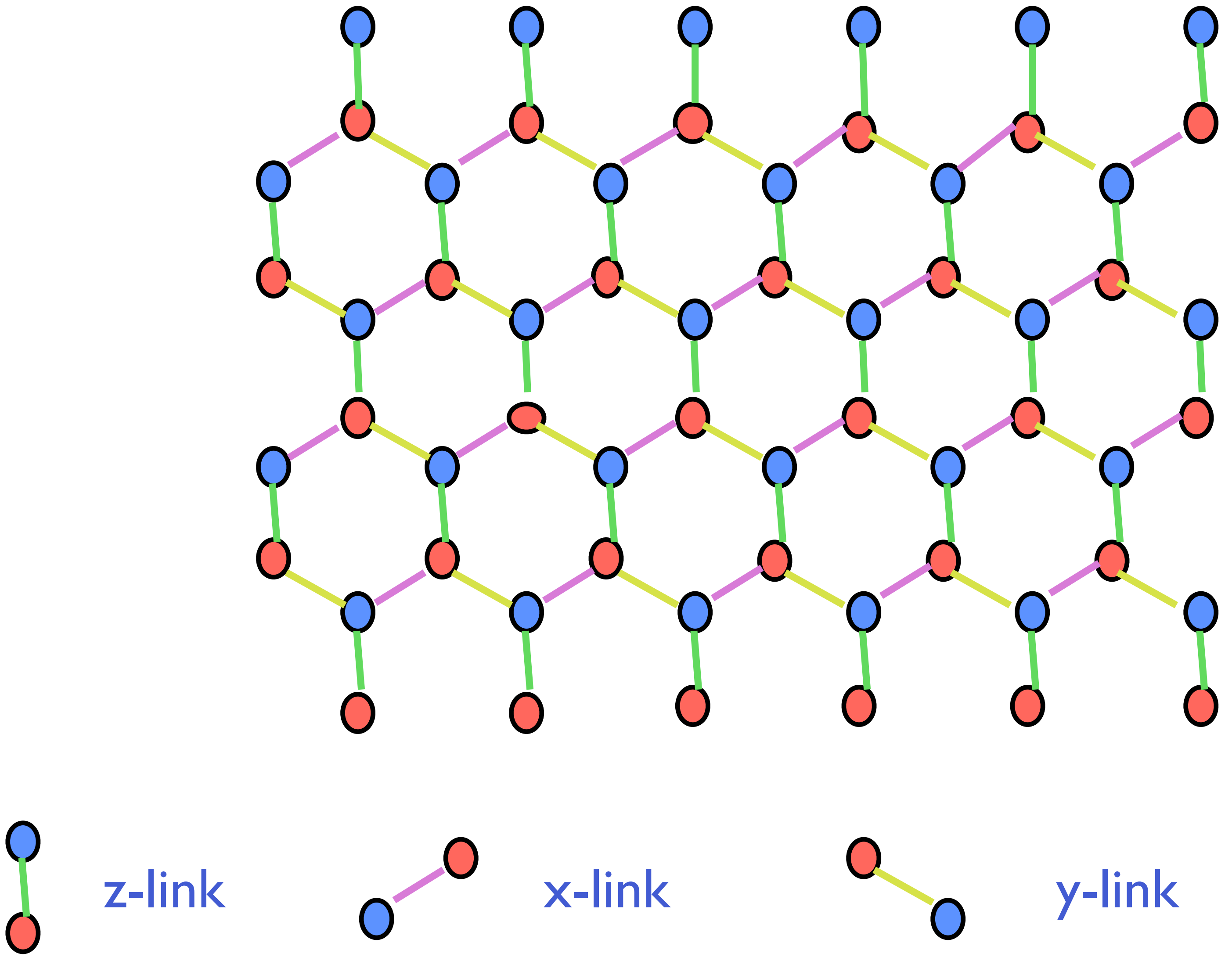
- We are analyzing a model with spins on a honeycomb lattice and the Hamiltonian

$$H = -J_x \sum_{x \text{ links}} \sigma_j^x \sigma_k^x - J_y \sum_{y \text{ links}} \sigma_j^y \sigma_y^x - J_z \sum_{z \text{ links}} \sigma_j^z \sigma_k^z$$

- We found that in each sector - defined by values of the vorticity $W_p = \pm 1$ on each plaquette

$$W_p = \sigma_1^x \sigma_2^y \sigma_3^z \sigma_4^x \sigma_5^y \sigma_6^z$$

the Hamiltonian reduced to a free fermion Hamiltonian.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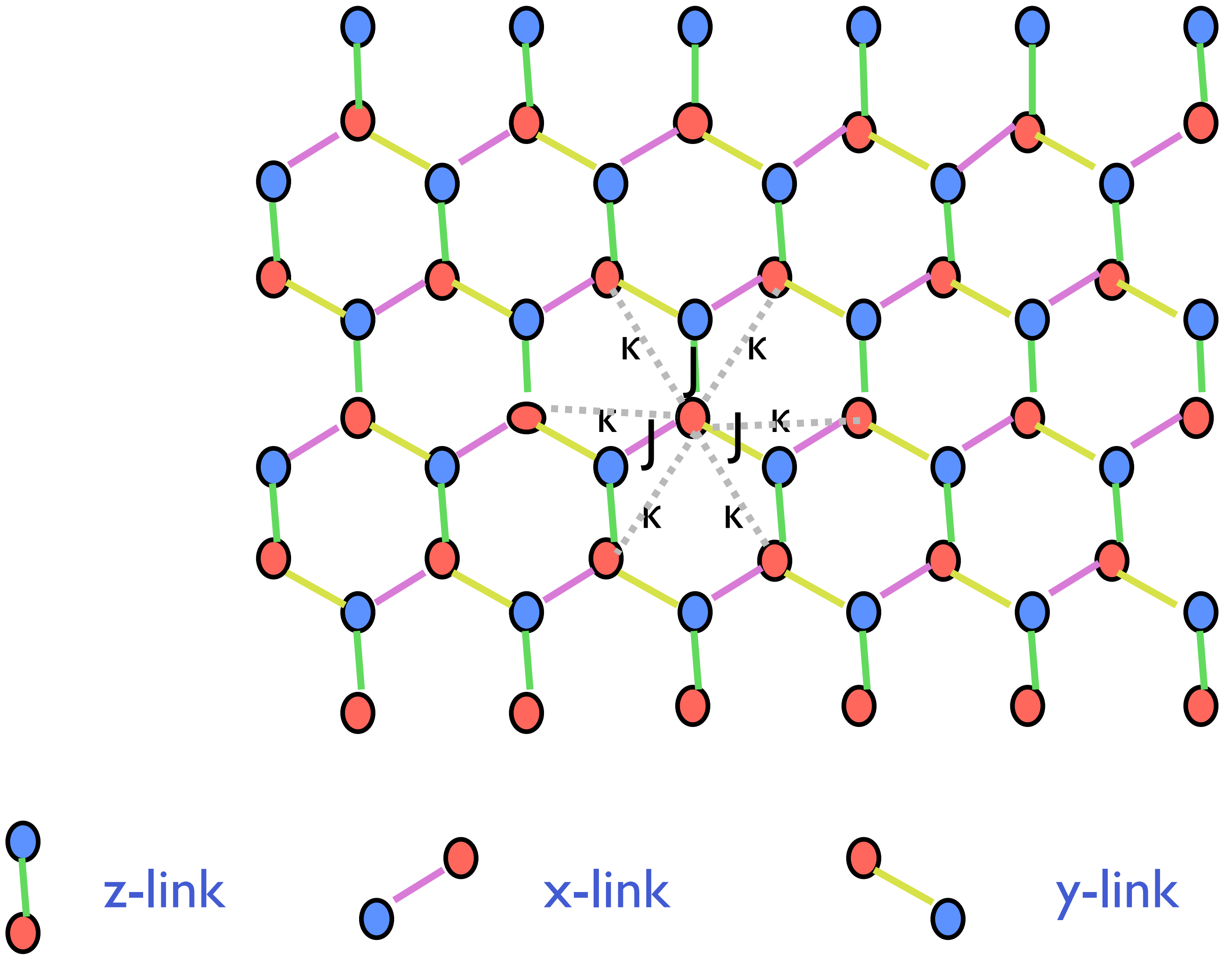
- The lowest-energy sector is the vortex free sector, with all $W_p = 1$.

- In that sector, the hopping amplitude is proportional to J, and connects nearest neighbors. Hops take us from even to odd (red and blue) sites.

- In the presence of a magnetic field, i.e. a perturbation

$$V = -\sum_j (h_x \sigma_j^x + h_y \sigma_j^y + h_z \sigma_j^z)$$

we have next-nearest-neighbor hopping amplitude with coefficient $\kappa \propto h_x h_y h_z / J^2$.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- Processing the original term:

$$H_0 \sim iJ \sum_{k,\mu} a_k a_{k+\mu}$$

$$\tilde{a}_{\pm}(q) = \frac{1}{\sqrt{V}} \sum e^{iqk} a_{k\pm}$$

$$a_{k\pm} = \frac{1}{\sqrt{V}} \sum e^{-iqk} \tilde{a}_{\pm}(q)$$

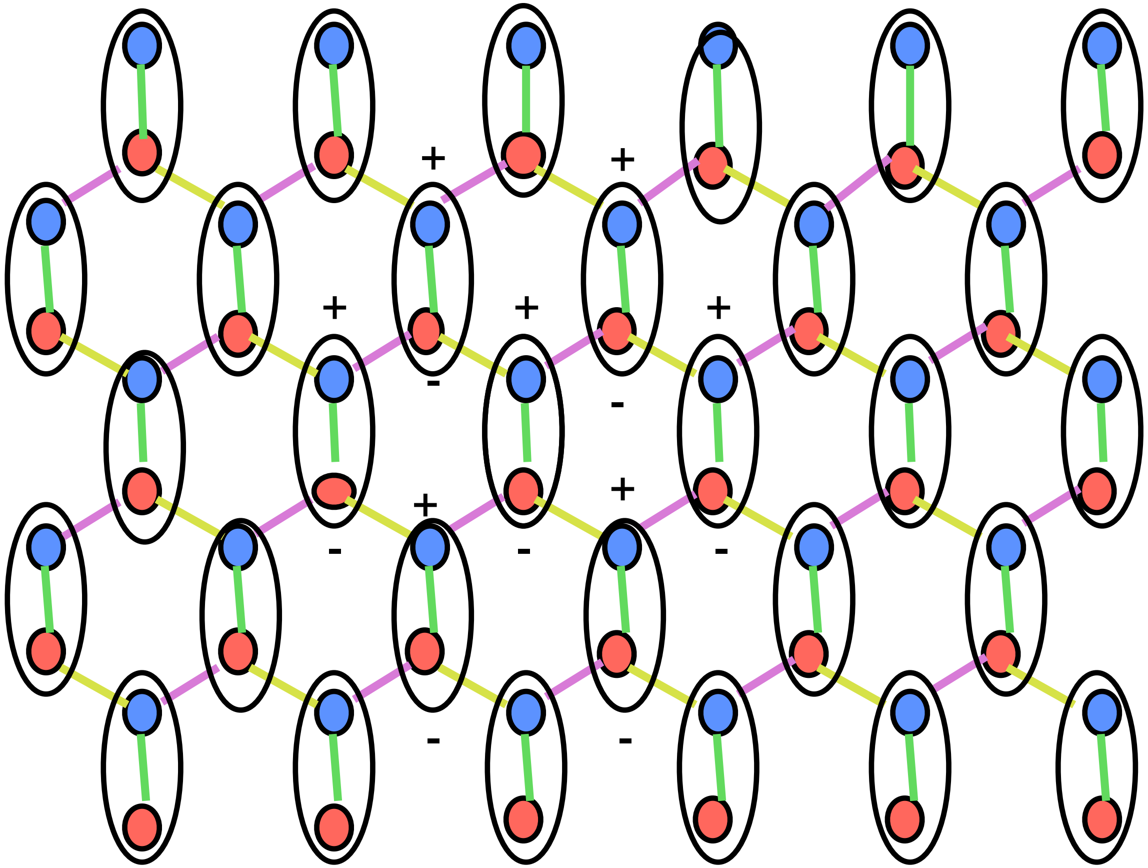
See unit cell, next slide

$$H_0 \sim i\frac{J}{V} \sum_{k,\mu,p,q} \tilde{a}(q)\tilde{a}(p) e^{-iqk} e^{-ip(k+\mu)}$$

$$\sim iJ \sum_{q,\mu} \tilde{a}_{\pm}(q)\tilde{a}_{\mp}(-q) e^{-ip\mu}$$

Sum on k

Even↔Odd



- Thus for each q we have the off-diagonal matrix

$$\begin{pmatrix} 0 & if(q) \\ -if(q)^* & 0 \end{pmatrix}$$

$$f(q) \sim J(e^{-iq_y} + e^{-iq_x\frac{\sqrt{3}}{2}+iq_y\frac{1}{2}} + e^{iq_x\frac{\sqrt{3}}{2}+iq_y\frac{1}{2}})$$

(This differs from Kitaev in an overall phase and scale.)

(In going from - to +, there's a conventional change in sign, and μ → -μ.)

- This leads to energies $\pm |f(q)|$.
- There is a zero at $q_y = 0$, $q_x = \pm(2/\sqrt{3})(2\pi/3)$

- A similar calculation gives for the perturbation term

$$\begin{pmatrix} \Delta(q) & 0 \\ 0 & -\Delta(q) \end{pmatrix}$$

where $\Delta(q^*) \sim h_x h_y h_z / J^2 \neq 0$ is real.

- The sum looks like $\begin{pmatrix} \Delta(q) & if(q) \\ -if(q) & -\Delta(q) \end{pmatrix}$. The 0 eigenvalue is lifted, since the determinant won't vanish.

- The separation of modes into positive and negative frequencies depends on the direction. To "straighten out" the gap, we must make a q-dependent rotation of the mode basis. There is significant topology in this collection of rotations, as we'll analyze next time. It leads to funny quantum numbers on vortices and to edge currents in response to weak perturbations.

- [Yesterday's Chez Pierre seminar concerned closely related issues potentially leading to dissipationless spin Hall effects. Very ~~hot~~ cool stuff!]

The Yoga of Quantum Number Flows

- Reference: "Fractional Quantum Numbers on Solitons", J. Goldstone and F. Wilczek, PRL 47 986 (1981)
- We've been led to topology of the "mass term" in momentum space. One can have closely related effects in real space, that are easier to visualize (and interesting in themselves).

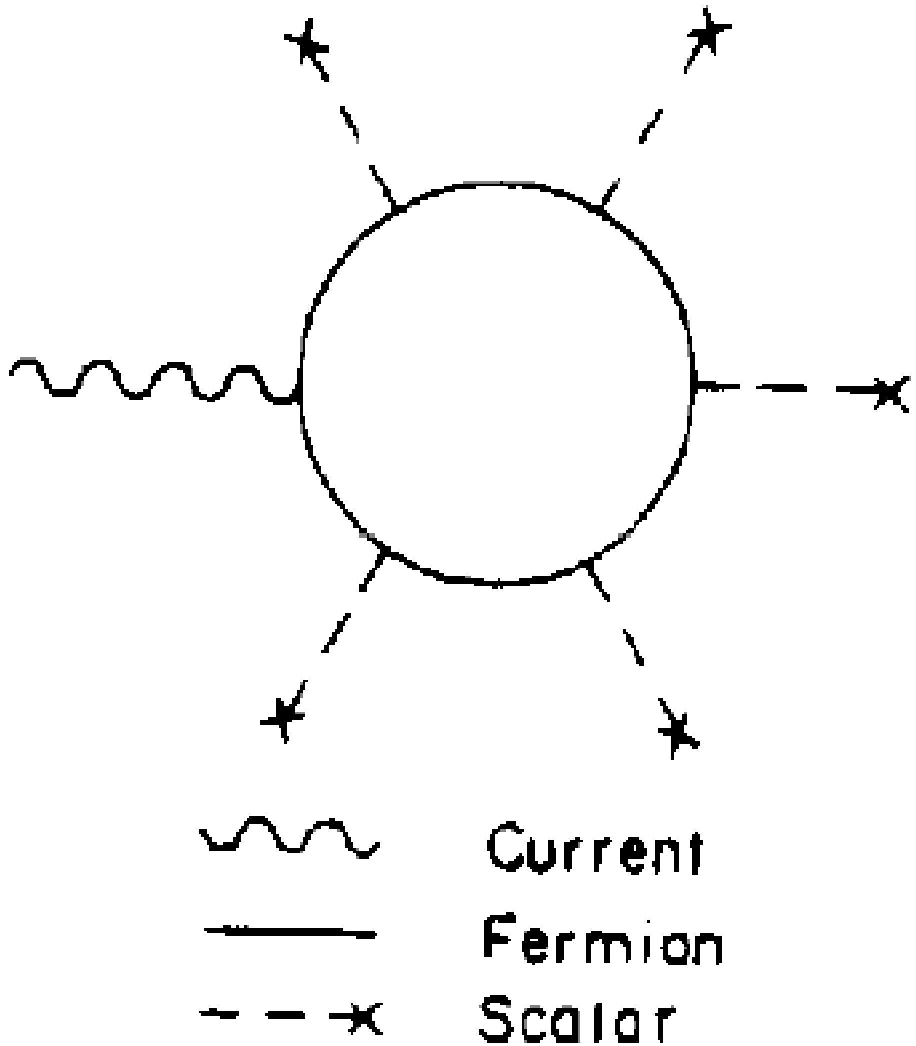
- Consider massless fermions in 1+1 dimensions interacting with two scalar fields Φ_1, Φ_2 according to $\mathcal{L}_I = g\bar{\psi}(\phi_1 + i\gamma_5\phi_2)\psi$.

- This is invariant under the chiral transformation

$$\begin{array}{rcl} \phi_1 & \rightarrow & \cos\theta\ \phi_1 + \sin\theta\ \phi_2 \\ \phi_2 & \rightarrow & -\sin\theta\ \phi_1 + \cos\theta\ \phi_2 \\ \psi & \rightarrow & e^{i\gamma_5 \frac{\theta}{2}}\psi \end{array}$$

- So it's easy to calculate the current that flows in response to slow gradients in the fields. We rotate the constant part to Φ_1, treat it as a mass, and rotate back!

- From the Feynman graph displayed, we get the current (in the no-particle state)


Current
Fermion
Scalar

$$\begin{aligned}\langle j^\mu \rangle &= \frac{1}{2\pi}\epsilon^{\mu\nu}\epsilon_{ab}\frac{\phi_a\partial_\nu\phi_b}{|\phi|^2} \\ &= \frac{1}{2\pi}\epsilon^{\mu\nu}\partial_\nu \arctan\frac{\phi_2}{\phi_1}\end{aligned}$$

Note: Manifestly conserved!

- We can find the charge of the fermi sea, induced by a soliton by fixing $\Phi_1 = m/g$ and letting Φ_2 vary from $-v$ at $x = -\infty$ to $+v$ at $x = +\infty$. The result is

$$Q = \frac{1}{\pi}\arctan\left(\frac{gv}{m}\right)$$

- In general, this is transcendental.
- The integer part is ambiguous, but that's as it should be.
- For $m \rightarrow 0$ we get $Q = \pm 1/2$. Also in this limit the background is charge-conjugation invariant. Interpretation: $\exists$ 0-mode, that can be either occupied or not.

- This strategy for computing flows and anomalous quantum numbers, by widening the field space and building up the state you want adiabatically, can be widely generalized. It's a very worthwhile project - it could clarify some calculations that presently seem quite obscure, and bring out additional consequences.

- Most "order parameter" fields are confined to finite samples. To see what this implies, interpret the current as an addition to the effective ~~Lagrangian~~ action:

$$\int \Delta\mathcal{L}\,\Theta(S)\;dxdt\ {}^{?} =^{?} \ \int j^{\mu} A_{\mu}\,\Theta(S)\;dxdt$$

- That's not gauge invariant, (even) if j is manifestly conserved, because you can't integrate by parts.

- However, suppose j has the form

$$j^\mu = \epsilon^{\mu\nu\cdots}\partial_\nu(\text{stuff})$$

- Then you get a satisfactory action by taking the Θ function insider the ∂.
- The price of this is that there's an additional term at the boundary, from where ∂ acts on Θ.   It can be interpreted as representing topological edge currents.